Supplement to The Computer Solution of Symmetric Homogeneous Triangle Inequalities

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Because of space constraints, the proof of several theorems from the paper [1] had to be omitted. The computer-generated proofs of these theorems are given below. See [1] for the notation and details of how the proofs were generated.

Theorem T2.

$$\sum \sin A \le \frac{3}{2}\sqrt{3} \tag{1}$$

$$\sum \sin^2 A \le \frac{9}{4} \tag{2}$$

$$\sum \sin A \ge \sum \sin 2A \tag{3}$$

$$\prod \sin A \le \frac{3}{8}\sqrt{3} \tag{4}$$

$$1 < \sum \cos A \tag{5}$$

$$\sum \cos A \le \frac{3}{2} \tag{6}$$

$$\frac{3}{4} \le \sum \cos^2 A \tag{7}$$

$$\sum \cos^2 A < 3 \tag{8}$$

$$\sum \cos A \cos B \le \frac{3}{4} \tag{9}$$

$$\prod \cos A \le \frac{1}{8} \tag{10}$$

$$\prod \cos A \le \frac{1}{24} \sum \cos^2(A-B) \tag{11}$$

$$\sum \cot A \ge \sqrt{3} \tag{12}$$

$$\sum \cot^2 A \ge 1 \tag{13}$$

$$\sum \csc A \ge 2\sqrt{3} \tag{14}$$

$$\sum \csc^2 A \ge 4 \tag{15}$$

$$\frac{1 + \prod \cos A}{\prod \sin A} \ge \sqrt{3} \tag{16}$$

$$2\sum \cot A \ge \sum \csc A \tag{17}$$

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Computer Proof of Theorem T2.
(1) is equivalent to $27[4,2,0]+27[3,3,0] \geq 5[4,1,1]+30[3,2,1]+19[2,2,2]$ which follows from the following majorizations:

$$5[4,2,0] \geq 5[4,1,1]$$

$$22[4,2,0] \geq 22[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$19[3,3,0] \geq 19[2,2,2].$$

(2) is equivalent to $9[4,2,0]+9[3,3,0] \geq 7[4,1,1]+10[3,2,1]+[2,2,2]$ which follows from the following majorizations:

$$7[4,2,0] \geq 7[4,1,1]$$

$$2[4,2,0] \geq 2[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$[3,3,0] \geq [2,2,2].$$

(3) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(4) is equivalent to $27[8,4,0]+108[8,3,1]+81[8,2,2]+108[7,5,0]+540[7,4,1]+1080[7,3,2]+81[6,6,0]+1080[6,5,1]+2862[6,4,2]+1944[5,5,2] \geq 104[6,3,3]+5268[5,4,3]+2539[4,4,4]$ which follows from the following majorizations:

$$27[8,4,0] \geq 27[6,3,3]$$

$$77[8,3,1] \geq 77[6,3,3]$$

$$31[8,3,1] \geq 31[5,4,3]$$

$$81[8,2,2] \geq 81[5,4,3]$$

$$108[7,5,0] \geq 108[5,4,3]$$

$$540[7,4,1] \geq 540[5,4,3]$$

$$1080[7,3,2] \geq 1080[5,4,3]$$

$$81[6,6,0] \geq 81[5,4,3]$$

$$1080[6,5,1] \geq 1080[5,4,3]$$

$$2298[6,4,2] \geq 2298[5,4,3]$$

$$595[6,4,2] \geq 595[4,4,4]$$

$$1944[5,5,2] \geq 1944[4,4,4].$$

(5) is equivalent to $8xyz > 0$ which follows because a sum of positive terms is positive.
(6) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(7) is equivalent to $9[4,2,0]+9[3,3,0] \geq 7[4,1,1]+10[3,2,1]+[2,2,2]$ which follows from the following majorizations:

$$7[4,2,0] \geq 7[4,1,1]$$

$$2[4,2,0] \geq 2[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$[3,3,0] \geq [2,2,2].$$

(8) is equivalent to $32\sum x^4yz+64\sum! \; x^3y^2z+96x^2y^2z^2 > 0$ which is true because a sum of positive terms is positive.

(9) is equivalent to $7[4,2,0]+7[3,3,0] \geq [4,1,1]+6[3,2,1]+7[2,2,2]$ which follows from the following majorizations:

$$[4,2,0] \geq [4,1,1]$$

$$6[4,2,0] \geq 6[3,2,1]$$

$$7[3,3,0] \geq 7[2,2,2].$$

(10) is equivalent to $9[4,2,0]+9[3,3,0] \geq 7[4,1,1]+10[3,2,1]+[2,2,2]$ which follows from the following majorizations:

$$7[4,2,0] \geq 7[4,1,1]$$

$$2[4,2,0] \geq 2[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$[3,3,0] \geq [2,2,2].$$

(11) is equivalent to $27[8,4,0]+108[7,5,0]+60[7,4,1]+81[6,6,0]+240[6,5,1]+5[4,4,4] \geq 12[8,3,1]+7[8,2,2]+104[7,3,2]+106[6,4,2]+120[6,3,3]+16[5,5,2]+156[5,4,3]$ which follows from the following majorizations:

$$5[6,6,0]+5[4,4,4] \geq 10[6,4,2]$$

$$12[8,4,0] \geq 12[8,3,1]$$

$$7[8,4,0] \geq 7[8,2,2]$$

$$8[8,4,0] \geq 8[7,3,2]$$

$$96[7,5,0] \geq 96[7,3,2]$$

$$12[7,5,0] \geq 12[6,4,2]$$

$$60[7,4,1] \geq 60[6,4,2]$$

$$24[6,6,0] \geq 24[6,4,2]$$

$$52[6,6,0] \geq 52[6,3,3]$$

$$68[6,5,1] \geq 68[6,3,3]$$

$$16[6,5,1] \geq 16[5,5,2]$$

$$156[6,5,1] \geq 156[5,4,3].$$

(12) is equivalent to $[4,0,0]+4[3,1,0]+3[2,2,0] \geq 8[2,1,1]$ which follows from the following majorizations:

$$3[2,2,0] \geq 3[2,1,1]$$

$$4[3,1,0] \geq 4[2,1,1]$$

$$[4,0,0] \geq [2,1,1].$$

(13) is equivalent to $[4,0,0]+4[3,1,0]+3[2,2,0] \geq 8[2,1,1]$ which follows from the following majorizations:

$$3[2,2,0] \geq 3[2,1,1]$$

$$4[3,1,0] \geq 4[2,1,1]$$

$$[4,0,0] \geq [2,1,1].$$

(14) is equivalent to $[4,0,0]+12[3,1,0]+11[2,2,0] \geq 24[2,1,1]$ which follows from the following majorizations:

$$11[2,2,0] \geq 11[2,1,1]$$

$$12[3,1,0] \geq 12[2,1,1]$$
$$[4,0,0] \geq [2,1,1].$$

(15) is equivalent to $[4,0,0]+4[3,1,0]+3[2,2,0] \geq 8[2,1,1]$ which follows from the following majorizations:

$$3[2,2,0] \geq 3[2,1,1]$$
$$4[3,1,0] \geq 4[2,1,1]$$
$$[4,0,0] \geq [2,1,1].$$

(16) is equivalent to $[6,0,0]+8[5,1,0]+16[4,2,0]+2[4,1,1]+10[3,3,0] \geq 24[3,2,1]+13[2,2,2]$ which follows from the following majorizations:

$$[6,0,0] \geq [3,2,1]$$
$$8[5,1,0] \geq 8[3,2,1]$$
$$15[4,2,0] \geq 15[3,2,1]$$
$$[4,2,0] \geq [2,2,2]$$
$$2[4,1,1] \geq 2[2,2,2]$$
$$10[3,3,0] \geq 10[2,2,2].$$

(17) is equivalent to $[2,0,0] \geq [1,1,0]$ which follows from Muirhead's Theorem.

Theorem T4.

$$s^2 \geq 3K\sqrt{3} \tag{1}$$
$$s^2 \geq 3K\sqrt{3} + \frac{1}{2}\sum(a-b)^2 \tag{2}$$
$$\sum a^2 \geq 4K\sqrt{3} \tag{3}$$
$$\sum ab \geq 4K\sqrt{3} \tag{4}$$
$$\sum ab \geq 4K\sqrt{3} + \frac{1}{2}\sum(a-b)^2 \tag{5}$$
$$4K\sqrt{3} + \sum(a-b)^2 \leq \sum a^2 \tag{6}$$
$$\sum a^2 \leq 4K\sqrt{3} + 3\sum(a-b)^2 \tag{7}$$
$$12K\sqrt{3} + 2\sum(a-b)^2 \leq \left(\sum a\right)^2 \tag{8}$$
$$\left(\sum a\right)^2 \leq 12K\sqrt{3} + 8\sum(a-b)^2 \tag{9}$$
$$\sum a^4 \geq 16K^2 \tag{10}$$
$$\sum a^4 \geq 16K^2 + 4K\sqrt{3}\sum(a-b)^2 + \frac{1}{2}\left(\sum(a-b)^2\right)^2 \tag{11}$$
$$\sum a^2b^2 \geq 16K^2 \tag{12}$$
$$4K\sqrt{3} \leq \frac{9abc}{\sum a} \tag{13}$$
$$(abc)^2 \geq (\frac{4K}{\sqrt{3}})^3 \tag{14}$$
$$\frac{1}{12}(\sum ab - \frac{1}{2}\sum a^2)(3\sum ab - \frac{5}{2}\sum a^2) \leq K^2 \tag{15}$$
$$K^2 \leq \frac{1}{12}(\sum ab - \frac{1}{2}\sum a^2)^2 \tag{16}$$
$$27\prod(b^2+c^2-a^2)^2 \leq (4K)^6 \tag{17}$$

Computer Proof of Theorem T4.
(1) is equivalent to $[3,0,0]+6[2,1,0] \geq 7[1,1,1]$ which follows from the following majorizations:

$$[3,0,0] \succ [1,1,1]$$

$$6[2,1,0] \succ 6[1,1,1].$$

(2) is equivalent to $[2,2,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(3) is equivalent to $[4,0,0]+4[3,1,0]+3[2,2,0] \geq 8[2,1,1]$ which follows from the following majorizations:

$$[4,0,0] \succ [2,1,1]$$

$$4[3,1,0] \succ 4[2,1,1]$$

$$3[2,2,0] \succ 3[2,1,1].$$

(4) is equivalent to $[4,0,0]+12[3,1,0]+11[2,2,0] \geq 24[2,1,1]$ which follows from the following majorizations:

$$[4,0,0] \succ [2,1,1]$$

$$12[3,1,0] \succ 12[2,1,1]$$

$$11[2,2,0] \succ 11[2,1,1].$$

(5) is equivalent to $[2,2,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(6) is equivalent to $[2,2,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(7) could not be handled by Algorithm K.
(8) is equivalent to $[2,2,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(9) could not be handled by Algorithm K.
(10) is equivalent to $[4,0,0]+4[3,1,0]+3[2,2,0] \geq 8[2,1,1]$ which follows from the following majorizations:

$$[4,0,0] \succ [2,1,1]$$

$$4[3,1,0] \succ 4[2,1,1]$$

$$3[2,2,0] \succ 3[2,1,1].$$

(11) is equivalent to $2[6,2,0]+2[4,4,0]+8[4,2,2]+[3,3,2] \geq [6,1,1]+2[5,2,1]+10[4,3,1]$ which Algorithm K could not prove automatically.
(12) is equivalent to $[4,0,0]+4[3,1,0]+3[2,2,0] \geq 8[2,1,1]$ which follows from the following majorizations:

$$[4,0,0] \succ [2,1,1]$$

$$4[3,1,0] \succ 4[2,1,1]$$

$$3[2,2,0] \succ 3[2,1,1].$$

(13) is equivalent to $27[4,2,0]+27[3,3,0] \geq 5[4,1,1]+30[3,2,1]+19[2,2,2]$ which follows from the following majorizations:

$$5[4,2,0] \succ 5[4,1,1]$$

$$22[4,2,0] \succ 22[3,2,1]$$

$$8[3,3,0] \succ 8[3,2,1]$$

$$19[3,3,0] \succ 19[2,2,2].$$

(14) is equivalent to $3[4,2,0]+3[4,1,1]+3[3,3,0]+18[3,2,1]+5[2,2,2] \geq 32[2,1,1]$ which follows from the following majorizations:

$$3[4,2,0] \succ 3[2,1,1]$$

$$3[4,1,1] \succ 3[2,1,1]$$

$$3[3,3,0] \succ 3[2,1,1]$$

$$18[3,2,1] \succ 18[2,1,1]$$

$$5[2,2,2] \succ 5[2,1,1].$$

(15) is equivalent to $[3,1,0] \geq [2,2,0]$ which follows from Muirhead's Theorem.
(16) is equivalent to $[2,2,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(17) is equivalent to $108[8,3,1] + 324[7,4,1] + 432[6,5,1] + 19[4,4,4] \geq 27[8,4,0] + 81[8,2,2] + 108[7,5,0] + 216[7,3,2] + 81[6,6,0] + 54[6,4,2] + 184[6,3,3] + 132[5,4,3]$ which Algorithm K could not prove automatically.

Theorem T5.

$$2r \leq R \tag{1}$$

$$\sum a \leq 3R\sqrt{3} \tag{2}$$

$$s \leq 2R + (3\sqrt{3} - 4)r \tag{3}$$

$$9r(4R + r) \leq 3s^2 \tag{4}$$

$$3s^2 \leq (4R + r)^2 \tag{5}$$

$$6r(4R + r) \leq 2s^2 \tag{6}$$

$$2s^2 \leq 2(2R + r)^2 + R^2 \tag{7}$$

$$2s^2(2R - r) \leq R(4R + r)^2 \tag{8}$$

$$r(16R - 5r) \leq s^2 \tag{9}$$

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \tag{10}$$

$$s^2 \geq 27r^2 \tag{11}$$

$$2s^2 \geq 27Rr \tag{12}$$

$$36r^2 \leq \sum a^2 \tag{13}$$

$$\sum a^2 \leq 9R^2 \tag{14}$$

$$24Rr - 12r^2 \leq \sum a^2 \tag{15}$$

$$\sum a^2 \leq 8R^2 + 4r^2 \tag{16}$$

$$36r^2 \leq \sum ab \tag{17}$$

$$\sum ab \leq 9R^2 \tag{18}$$

$$4r(5R - r) \leq \sum ab \tag{19}$$

$$\sum ab \leq 4(R + r)^2 \tag{20}$$

$$36r^2 \leq 4r(5R - r) \tag{21}$$

$$4r(5R - r) \leq \sum ab \tag{22}$$

$$4(R + r)^2 \leq 9R^2 \tag{23}$$

$$\sum a(s - a)) \leq 9Rr \tag{24}$$

$$abc \leq 8R^2r + (12\sqrt{3} - 16)Rr^2 \tag{25}$$

$$\frac{\sqrt{3}}{R} \leq \sum \frac{1}{a} \tag{26}$$

$$\sum \frac{1}{a} \leq \frac{\sqrt{3}}{2r} \tag{27}$$

$$\frac{3\sqrt{3}}{2(R+r)} \leq \sum \frac{1}{a} \tag{28}$$

$$\frac{1}{R^2} \leq \sum \frac{1}{ab} \tag{29}$$

$$\sum \frac{1}{ab} \leq \frac{1}{4r^2} \tag{30}$$

$$8r(R - 2r) \leq \sum (a-b)^2 \tag{31}$$

$$\sum (a-b)^2 \leq 8R(R - 2r) \tag{32}$$

$$4r^2 \leq \frac{abc}{\sum a} \tag{33}$$

$$abc \leq (R\sqrt{3})^3 \tag{34}$$

$$5R - r \geq s\sqrt{3} \tag{35}$$

$$54Rr \leq 3\sum ab \tag{36}$$

Computer Proof of Theorem T5.

(1) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.

(2) is equivalent to $27[4,2,0] + 27[3,3,0] \geq 5[4,1,1] + 30[3,2,1] + 19[2,2,2]$ which follows from the following majorizations:

$$5[4,2,0] \geq 5[4,1,1]$$

$$22[4,2,0] \geq 22[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$19[3,3,0] \geq 19[2,2,2].$$

(3) could not be handled because of a bug in the implementation of Algorithm K.

(4) is equivalent to $[3,0,0]+6[2,1,0]+2[1,1,1] \geq 18[1,1,0]$ which algorithm K could not prove automatically.

(5) is equivalent to $2[6,2,0] + 8[5,3,0] + 6[4,4,0] + 4[4,3,1] \geq [6,1,1] + 2[5,2,1] + 7[4,2,2] + 10[3,3,2]$ which follows from the following majorizations:

$$[6,2,0] \geq [6,1,1]$$

$$[6,2,0] \geq [5,2,1]$$

$$[5,3,0] \geq [5,2,1]$$

$$7[5,3,0] \geq 7[4,2,2]$$

$$6[4,4,0] \geq 6[3,3,2]$$

$$4[4,3,1] \geq 4[3,3,2].$$

(6) is equivalent to $[3,0,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.

(7) is equivalent to $9[6,2,0]+36[5,3,0]+27[4,4,0]+10[4,3,1] \geq 7[6,1,1]+20[5,2,1]+28[4,2,2]+27[3,3,2]$ which follows from the following majorizations:

$$7[6,2,0] \geq 7[6,1,1]$$

$$2[6,2,0] \geq 2[5,2,1]$$

$$18[5,3,0] \geq 18[5,2,1]$$

$$18[5,3,0] \geq 18[4,2,2]$$

$$10[4,4,0] \geq 10[4,2,2]$$

$$17[4,4,0] \geq 17[3,3,2]$$

$$10[4,3,1] \geq 10[3,3,2].$$

(8) is equivalent to $[8,3,0]+5[7,4,0]+10[6,5,0]+[6,4,1]+2[5,5,1]+5[4,4,3] \geq [8,2,1]+2[7,3,1]+3[7,2,2]+10[6,3,2]+7[5,4,2]+[5,3,3]$ which algorithm K could not prove automatically.
(9) is equivalent to $[4,0,0]+[2,1,1] \geq 2[2,2,0]$ which follows from the following majorizations:

$$[4,0,0]+[2,1,1] \geq 2[3,1,0]$$

$$2[3,1,0] \geq 2[2,2,0].$$

(10) is equivalent to $[4,2,0]+[3,3,0]+[2,2,2] \geq [4,1,1]+2[3,2,1]$ which follows from the following majorizations:

$$[3,3,0]+[2,2,2] \geq 2[3,2,1]$$

$$[4,2,0] \geq [4,1,1].$$

(11) is equivalent to $[3,0,0]+6[2,1,0] \geq 7[1,1,1]$ which follows from the following majorizations:

$$6[2,1,0] \geq 6[1,1,1]$$

$$[3,0,0] \geq [1,1,1].$$

(12) is equivalent to $4[3,0,0] \geq 3[2,1,0]+[1,1,1]$ which follows from the following majorizations:

$$3[3,0,0] \geq 3[2,1,0]$$

$$[3,0,0] \geq [1,1,1].$$

(13) is equivalent to $[3,0,0]+4[2,1,0] \geq 5[1,1,1]$ which follows from the following majorizations:

$$4[2,1,0] \geq 4[1,1,1]$$

$$[3,0,0] \geq [1,1,1].$$

(14) is equivalent to $9[4,2,0]+9[3,3,0] \geq 7[4,1,1]+10[3,2,1]+[2,2,2]$ which follows from the following majorizations:

$$7[4,2,0] \geq 7[4,1,1]$$

$$2[4,2,0] \geq 2[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$[3,3,0] \geq [2,2,2].$$

(15) is equivalent to $[4,0,0]+[2,1,1]\geq 2[2,2,0]$ which follows from the following majorizations:

$$[4,0,0]+[2,1,1]\geq 2[3,1,0]$$

$$2[3,1,0]\geq 2[2,2,0].$$

(16) is equivalent to $[4,2,0]+[3,3,0]+[2,2,2]\geq [4,1,1]+2[3,2,1]$ which follows from the following majorizations:

$$[3,3,0]+[2,2,2]\geq 2[3,2,1]$$

$$[4,2,0]\geq [4,1,1].$$

(17) is equivalent to $[3,0,0]+8[2,1,0]\geq 9[1,1,1]$ which follows from the following majorizations:

$$8[2,1,0]\geq 8[1,1,1]$$

$$[3,0,0]\geq [1,1,1].$$

(18) is equivalent to $9[4,2,0]+[4,1,1]+9[3,3,0]\geq 10[3,2,1]+9[2,2,2]$ which follows from the following majorizations:

$$9[4,2,0]\geq 9[3,2,1]$$

$$[4,1,1]\geq [3,2,1]$$

$$9[3,3,0]\geq 9[2,2,2].$$

(19) is equivalent to $[4,0,0]+[2,1,1]\geq 2[2,2,0]$ which follows from the following majorizations:

$$[4,0,0]+[2,1,1]\geq 2[3,1,0]$$

$$2[3,1,0]\geq 2[2,2,0].$$

(20) is equivalent to $[4,2,0]+[3,3,0]+[2,2,2]\geq [4,1,1]+2[3,2,1]$ which follows from the following majorizations:

$$[3,3,0]+[2,2,2]\geq 2[3,2,1]$$

$$[4,2,0]\geq [4,1,1].$$

(21) is equivalent to $[2,1,0]\geq [1,1,1]$ which follows from Muirhead's Theorem.
(22) is equivalent to $[4,0,0]+[2,1,1]\geq 2[2,2,0]$ which follows from the following majorizations:

$$[4,0,0]+[2,1,1]\geq 2[3,1,0]$$

$$2[3,1,0]\geq 2[2,2,0].$$

(23) is equivalent to $5[6,2,0]+5[6,1,1]+20[5,3,0]+28[5,2,1]+15[4,4,0]+34[4,3,1]\geq 28[4,2,2]+79[3,3,2]$ which follows from the following majorizations:

$$5[6,2,0]\geq 5[4,2,2]$$

$$5[6,1,1]\geq 5[4,2,2]$$

$$18[5,3,0]\geq 18[4,2,2]$$

$$2[5,3,0]\geq 2[3,3,2]$$

$$28[5,2,1]\geq 28[3,3,2]$$

$$15[4,4,0]\geq 15[3,3,2]$$

$$34[4,3,1]\geq 34[3,3,2].$$

(24) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(25) could not be handled because of a bug in the implementation of Algorithm K.
(26) is equivalent to $[4,0,0]+12[3,1,0]+11[2,2,0] \geq 24[2,1,1]$ which follows from the following majorizations:

$$11[2,2,0] \geq 11[2,1,1]$$

$$12[3,1,0] \geq 12[2,1,1]$$

$$[4,0,0] \geq [2,1,1].$$

(27) is equivalent to $3[5,2,0] + [5,1,1] + 9[4,3,0] + 3[4,2,1] \geq [3,3,1] + 15[3,2,2]$ which follows from the following majorizations:

$$[5,2,0] \geq [3,3,1]$$

$$2[5,2,0] \geq 2[3,2,2]$$

$$[5,1,1] \geq [3,2,2]$$

$$9[4,3,0] \geq 9[3,2,2]$$

$$3[4,2,1] \geq 3[3,2,2].$$

(28) is equivalent to $[10,2,0]+[10,1,1]+10[9,3,0]+38[9,2,1]+41[8,4,0]+132[8,3,1]+99[8,2,2]+90[7,5,0]+198[7,4,1]+296[7,3,2]+58[6,6,0]+182[6,5,1]+18[6,4,2] \geq 32[6,3,3]+117[5,5,2]+798[5,4,3]+217[4,4,4]$ which follows from the following majorizations:

$$[10,2,0] \geq [6,3,3]$$

$$[10,1,1] \geq [6,3,3]$$

$$10[9,3,0] \geq 10[6,3,3]$$

$$20[9,2,1] \geq 20[6,3,3]$$

$$18[9,2,1] \geq 18[5,5,2]$$

$$41[8,4,0] \geq 41[5,5,2]$$

$$58[8,3,1] \geq 58[5,5,2]$$

$$74[8,3,1] \geq 74[5,4,3]$$

$$99[8,2,2] \geq 99[5,4,3]$$

$$90[7,5,0] \geq 90[5,4,3]$$

$$198[7,4,1] \geq 198[5,4,3]$$

$$296[7,3,2] \geq 296[5,4,3]$$

$$41[6,6,0] \geq 41[5,4,3]$$

$$17[6,6,0] \geq 17[4,4,4]$$

$$182[6,5,1] \geq 182[4,4,4]$$

$$18[6,4,2] \geq 18[4,4,4].$$

(29) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(30) is equivalent to $[4,1,0] + 3[3,2,0] \geq [3,1,1] + 3[2,2,1]$ which follows from the following majorizations:

$$[4,1,0] \geq [3,1,1]$$

$$3[3,2,0] \geq 3[2,2,1].$$

(31) is equivalent to $[4,0,0]+[2,1,1] \geq 2[2,2,0]$ which follows from the following majorizations:

$$[4,0,0]+[2,1,1] \geq 2[3,1,0]$$

$$2[3,1,0] \geq 2[2,2,0].$$

(32) is equivalent to $[4,2,0]+[3,3,0]+[2,2,2] \geq [4,1,1]+2[3,2,1]$ which follows from the following majorizations:

$$[3,3,0]+[2,2,2] \geq 2[3,2,1]$$

$$[4,2,0] \geq [4,1,1].$$

(33) is equivalent to $[4,1,0]+3[3,2,0] \geq [3,1,1]+3[2,2,1]$ which follows from the following majorizations:

$$[4,1,0] \geq [3,1,1]$$

$$3[3,2,0] \geq 3[2,2,1].$$

(34) is equivalent to $27[8,4,0]+108[8,3,1]+81[8,2,2]+108[7,5,0]+540[7,4,1]+1080[7,3,2]+81[6,6,0]+1080[6,5,1]+2862[6,4,2]+1944[5,5,2] \geq 104[6,3,3]+5268[5,4,3]+2539[4,4,4]$ which follows from the following majorizations:

$$27[8,4,0] \geq 27[6,3,3]$$

$$77[8,3,1] \geq 77[6,3,3]$$

$$31[8,3,1] \geq 31[5,4,3]$$

$$81[8,2,2] \geq 81[5,4,3]$$

$$108[7,5,0] \geq 108[5,4,3]$$

$$540[7,4,1] \geq 540[5,4,3]$$

$$1080[7,3,2] \geq 1080[5,4,3]$$

$$81[6,6,0] \geq 81[5,4,3]$$

$$1080[6,5,1] \geq 1080[5,4,3]$$

$$2267[6,4,2] \geq 2267[5,4,3]$$

$$595[6,4,2] \geq 595[4,4,4]$$

$$1944[5,5,2] \geq 1944[4,4,4].$$

(35) is equivalent to $25[6,2,0]+[6,1,1]+100[5,3,0]+20[5,2,1]+75[4,4,0]+50[4,3,1] \geq 92[4,2,2]+179[3,3,2]$ which follows from the following majorizations:

$$25[6,2,0] \geq 25[4,2,2]$$

$$[6,1,1] \geq [4,2,2]$$

$$66[5,3,0] \geq 66[4,2,2]$$

$$34[5,3,0] \geq 34[3,3,2]$$

$$20[5,2,1] \geq 20[3,3,2]$$

$$75[4,4,0] \geq 75[3,3,2]$$

$$50[4,3,1] \geq 50[3,3,2].$$

(36) is equivalent to $[3,0,0] \geq [2,1,0]$ which follows from Muirhead's Theorem.

Theorem T6.

$$2\sum h_a \le \sqrt{3}\sum a \tag{1}$$

$$3\sum ab \ge \sum h_a h_b \tag{2}$$

$$\sum a^3 > \frac{8}{7}\sum h_a^3 \tag{3}$$

$$\sum \frac{a^2}{h_b^2+h_c^2} \ge 2 \tag{4}$$

$$\sum h_a \ge 9r \tag{5}$$

$$\sum h_a \le 3(R+r) \tag{6}$$

$$\sum h_a \le 2R+5r \tag{7}$$

$$\frac{2r(5R-r)}{R} \le \sum h_a \tag{8}$$

$$\sum h_a \le \frac{2(R+r)^2}{R} \tag{9}$$

$$2\sum h_a h_b \le 6K\sqrt{3} \tag{10}$$

$$6K\sqrt{3} \le 27Rr \tag{11}$$

$$\prod h_a \ge 27r^3 \tag{12}$$

$$\sum \frac{1}{h_a-2r} \ge \frac{3}{r} \tag{13}$$

$$\sum \frac{h_a+r}{h_a-r} \ge 6 \tag{14}$$

Computer Proof of Theorem T6.
(1) is equivalent to $3[6,2,0]+[6,1,1]+12[5,3,0]+8[5,2,1]+9[4,4,0]+10[4,3,1] \ge 12[4,2,2]+31[3,3,2]$ which follows from the following majorizations:

$$3[6,2,0] \ge 3[4,2,2]$$

$$[6,1,1] \ge [4,2,2]$$

$$8[5,3,0] \ge 8[4,2,2]$$

$$4[5,3,0] \ge 4[3,3,2]$$

$$8[5,2,1] \ge 8[3,3,2]$$

$$9[4,4,0] \ge 9[3,3,2]$$

$$10[4,3,1] \ge 10[3,3,2].$$

(2) is equivalent to $6\sum! \, x^4y + 24\sum! \, x^3y^2 + 32\sum x^3yz + 52\sum x^2y^2z \ge 0$ which follows because a sum of positive terms is positive.

(3) is equivalent to

$$\begin{aligned}
&196[18,6,0] + 1176[18,5,1] + 2940[18,4,2] + 1764[17,7,0]\\
&+ 12348[17,6,1] + 37044[17,5,2] + 24876[17,4,3] + 7497[16,8,0]\\
&+ 59976[16,7,1] + 209916[16,6,2] + 272376[16,5,3] + 102651[16,4,4]\\
&+ 20188[15,9,0] + 181692[15,8,1] + 726768[15,7,2]\\
&+ 1335344[15,6,3] + 1265736[15,5,4] + 38808[14,10,0]\\
&+ 389256[14,9,1] + 1753416[14,8,2] + 4036800[14,7,3]\\
&+ 4936224[14,6,4] + 2271672[14,5,5] + 56448[13,11,0]\\
&+ 629748[13,10,1] + 3167556[13,9,2] + 8582304[13,8,3]\\
&+ 12977280[13,7,4] + 12716328[13,6,5] + 31899[12,12,0]\\
&+ 795564[12,11,1] + 4459392[12,10,2] + 13760220[12,9,3]\\
&+ 24732570[12,8,4] + 27555960[12,7,5] + 12815024[12,6,6]\\
&+ 2494296[11,11,2] + 17278752[11,10,3] + 35818764[11,9,4]\\
&+ 46177704[11,8,5] + 43019088[11,7,6] + 20209620[10,10,4]\\
&+ 59783100[10,9,5] + 59875008[10,8,6] + 27070128[10,7,7]\\
&+ 33566236[9,9,6] + 59149692[9,8,7] + 9236871[8,8,8]\\
&> 88[18,3,3]
\end{aligned}$$

which follows from the majorization:

$$88[18,4,2] \geq 88[18,3,3].$$

(4) is equivalent to $5[8,2,0]+20[7,3,0]+40[6,4,0]+28[6,3,1]+25[5,5,0]+78[5,4,1] \geq 3[8,1,1]+4[7,2,1]+20[6,2,2]+64[5,3,2]+25[4,4,2]+80[4,3,3]$ which follows from the following majorizations:

$$3[8,2,0] \geq 3[8,1,1]$$

$$2[8,2,0] \geq 2[7,2,1]$$

$$2[7,3,0] \geq 2[7,2,1]$$

$$18[7,3,0] \geq 18[6,2,2]$$

$$2[6,4,0] \geq 2[6,2,2]$$

$$38[6,4,0] \geq 38[5,3,2]$$

$$26[6,3,1] \geq 26[5,3,2]$$

$$2[6,3,1] \geq 2[4,4,2]$$

$$23[5,5,0] \geq 23[4,4,2]$$

$$2[5,5,0] \geq 2[4,3,3]$$

$$78[5,4,1] \geq 78[4,3,3].$$

(5) is equivalent to $2[6,0,0]+[4,1,1]+5[3,3,0]+[2,2,2] \geq 4[5,1,0]+3[4,2,0]+2[3,2,1]$ which Algorithm K could not prove automatically.

(6) is equivalent to $3[5,2,0]+9[4,3,0]+[3,3,1] \geq [5,1,1]+[4,2,1]+11[3,2,2]$ which follows from the following majorizations:

$$[5,2,0] \geq [5,1,1]$$

$$[5,2,0] \geq [4,2,1]$$

$$[5,2,0] \geq [3,2,2]$$

$$9[4,3,0] \geq 9[3,2,2]$$

$$[3,3,1] \geq [3,2,2].$$

(7) is equivalent to $[5,2,0]+3[4,3,0]+[3,3,1] \geq [5,1,1]+[4,2,1]+3[3,2,2]$ which follows from the following majorizations:

$$[5,2,0] \geq [5,1,1]$$

$$[4,3,0] \geq [4,2,1]$$

$$2[4,3,0] \geq 2[3,2,2]$$

$$[3,3,1] \geq [3,2,2].$$

(8) is equivalent to $[8,0,0]+2[6,1,1]+4[5,2,1]+6[4,4,0]+5[4,2,2] \geq 8[6,2,0]+8[4,3,1]+2[3,3,2]$ which Algorithm K could not prove automatically.
(9) is equivalent to $[4,2,0]+[3,3,0]+[2,2,2] \geq [4,1,1]+2[3,2,1]$ which follows from the following majorizations:

$$[3,3,0]+[2,2,2] \geq 2[3,2,1]$$

$$[4,2,0] \geq [4,1,1].$$

(10) is equivalent to $27[4,2,0]+27[3,3,0] \geq 5[4,1,1]+30[3,2,1]+19[2,2,2]$ which follows from the following majorizations:

$$5[4,2,0] \geq 5[4,1,1]$$

$$22[4,2,0] \geq 22[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$19[3,3,0] \geq 19[2,2,2].$$

(11) is equivalent to $27[4,2,0]+27[3,3,0] \geq 5[4,1,1]+30[3,2,1]+19[2,2,2]$ which follows from the following majorizations:

$$5[4,2,0] \geq 5[4,1,1]$$

$$22[4,2,0] \geq 22[3,2,1]$$

$$8[3,3,0] \geq 8[3,2,1]$$

$$19[3,3,0] \geq 19[2,2,2].$$

(12) is equivalent to $32[6,0,0]+73[3,3,0]+6[3,2,1]+15[2,2,2] \geq 48[5,1,0]+39[4,2,0]+39[4,1,1]$ which Algorithm K could not prove automatically.
(13) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's theorem.
(14) is equivalent to $[3,0,0]+[2,1,0] \geq 2[1,1,1]$ which follows from the following majorizations:

$$[2,1,0] \geq [1,1,1]$$

$$[3,0,0] \geq [1,1,1].$$

REFERENCE

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