On the Number of Lattice Points Inside a Convex Lattice n-gon

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A lattice point in the plane is a point with integer coordinates. A lattice polygon is a polygon whose vertices are all lattice points.

In this paper, we will investigate inequalities between the number of vertices, v, of a convex lattice polygon and the number, g, of lattice points in the interior of the polygon ("the interior lattice points"). If G denotes the number of lattice points inside or on the polygon, we will also discuss the relationship between v and G. If K is a set of points in the plane, then $G(K)$ denotes the number of lattice points int he set K.

A polygon with n vertices will be referred to as an n-gon.

In 1980, Arkinstall proved the Lattice Pentagon Theorem, which states that any convex lattice pentagon must contain an interior lattice point. We will investigate further similar relationships between v and g for lattice polygons.

To understand when two lattice polygons are "equivalent", we must first review some definitions concerning standard transformations of the plane. An affine transformation is a linear transformation followed by a translation. A unimodular transformation is one that preserves area. To be unimodular, the matrix corresponding to a linear transformation must have determinant ±1. If furthermore, the entries of the matrix are integers, then the transformation is known as an integral unimodular affine transformation. If f is an integral unimodular affine transformation, then f has the property that for any convex set, K, $G(f(K)) = G(K)$ (i.e. f preserves the number of lattice points in sets). An integral unimodular affine transformation (also known as an equiaffinity) in the plane can be expressed by the 3×3 matrix in the equation

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

where a, b, c, d, e, and f are integers and $|ad-bc| = 1$. This includes an integral translation by the vector (e, f).

Two lattice polygons are said to be *lattice equivalent* if one can be transformed into the other via an integral unimodular affine transformation.

1. Known results

Arkinstall [1] was the first to note that certain types of lattice polygons must necessarily have lattice points in their interior. For example, a convex lattice trapezium must contain an interior lattice point. (Recall that a *trapezium* is a quadrilateral with no two sides parallel.) We state below some of theorems that he proved which we will need to use later in this paper.

Reprinted from Congressus Numerantium 73(1990)99–124

The Lattice Trapezium Theorem (Arkinstall). *A convex lattice trapezium must contain an interior lattice point.*

The Lattice Pentagon Theorem (Arkinstall). *A convex lattice pentagon must contain an interior lattice point.*

This theorem will be used heavily in the remainder of this paper, so for completeness, we will reproduce Arkinstall's proof.

Figure 1-1

Proof. Let $ABCDE$ be a convex lattice pentagon. Since the sum of the interior angles of a pentagon is 3π, the sum of the 5 pairs of adjacent interior angles is 6π. Hence some pair of adjacent angles must sum to more than π. We may thus assume without loss of generality that $\angle A + \angle B > \pi$ (see figure 1-1). We may also assume that point C is not further from line AB than point E. Constructing parallelogram $ABCX$, we see that rays AX and CX lie inside angles A and C respectively. Thus point X lies inside the pentagon. But if three vertices of a parallelogram are lattice points, the fourth vertex must also be a lattice point. $\square$

The Central Hexagon Theorem (Arkinstall). *Let K be a convex lattice hexagon with precisely one interior lattice point. Then K is lattice equivalent to the centrally symmetric hexagon with vertices at $(1,0)$, $(1,1)$, $(0,1)$, $(-1,0)$, $(-1,-1)$, and $(0,-1)$.*

We will also occasional need to use the following result by Scott [5]. This result established a relationship between the number of lattice points on the boundary of a convex lattice polygon and the number of lattice points in its interior.

Theorem (Scott's Bound for b). *If a convex lattice polygon has g interior lattice points ($g > 0$) and b lattice points on its boundary, then $b \leq 2g + 7$. If $v > 3$, then $b \leq 2g + 6$.*

In 1989, Rabinowitz [4] catalogued all convex lattice polygons with at most one interior lattice point. The following two theorems (Census-0 and Census-1) summarize his results. They characterize those convex lattice polygons containing no lattice points and those containing precisely one interior lattice point, respectively.

Theorem (Census-0). *If K is a convex lattice polygon with no interior lattice points, then K is lattice equivalent to one of the following polygons:*

1. *the triangle whose vertices are $(0,0)$, $(2,0)$, and $(0,2)$*
2. *the triangle whose vertices are $(0,0)$, $(p,0)$, and $(0,1)$*
3. *the trapezoid whose vertices are $(0,0)$, $(p,0)$, $(0,1)$, and $(q,1)$*

   where p and q are any positive integers.

Theorem (Census-1). *If K is a convex lattice polygon with exactly one interior lattice point, then K is lattice equivalent to precisely one of the following 15 polygons:*

```
                                                  o . . .
o . . . .    o . . .     . o .     . . o      . . . .
. + . . .    . + . .     . + .     o + .      . + . .
o . . . o    o . . o     o . o     . o .      o . . o

. . o        . o .       o . o     o . o      o . . .
o + .        o + o       . + .     . + .      . + o .
o o .        . o .       o o .     o . o      o . . o

. o .        . o .       o o .     . o .      . o o
o + .        o + o       . + o     o + o      o + o
o . o        o . o       o . o     o o .      o o .
```

Figure 1-2

All convex lattice polygons with g=1

The dots represent lattice points and the circles represent the vertices of the polygon. The plus signs denote the interior lattice point.

2. Inequalities with v fixed

If we fix v, then g can get arbitrarily large. For given any polygon, K, with v vertices, we can expand it by any amount (by applying the transformation that maps (x,y) into (nx,ny) for some positive integer n). The resulting polygon has the same number of vertices, but g can get arbitrarily large. In other words, for a fixed v,

$$\sup\{g(K)|v(K)=v\}=\infty.$$

A more interesting problem is to find the minimum value that g can have when we fix v.

Notation. If K is a convex lattice polygon, then $g(K)$ denotes the number of lattice points in the interior of K, $b(K)$ denotes the number of lattice points on the boundary of K, and $G(K)=b(K)+g(K)$ denotes the number of lattice points inside or on K. Also, $v(K)$ represents the number of vertices of K. If K is implicit, these quantities may simply be referred to as g, b, G, and v, respectively.

The next two propositions follow immediately from Census-0.

Proposition 2.1. *A lattice triangle need not have any interior lattice points. We have $v = 3$ and $g = 0$ when and only when K is lattice equivalent to either*

1. *the triangle whose vertices are $(0, 0)$, $(2, 0)$, and $(0, 2)$ or*
2. *the triangle whose vertices are $(0, 0)$, $(p, 0)$, and $(0, 1)$*

   for some positive integer p.

Proposition 2.2. *If $v = 4$ then $g \geq 0$. Equality occurs when and only when K is lattice equivalent to the trapezoid whose vertices are $(0, 0)$, $(p, 0)$, $(0, 1)$, and $(q, 1)$ for some positive integers p and q.*

The situation for pentagons is determined by the Lattice Pentagon Theorem, ensuring that there is at least one interior lattice point. The equality case ($v = 5$ and $g = 1$) is determined by Census-1. This gives us the following proposition.

Proposition 2.3. *If $v = 5$ then $g \geq 1$. Equality holds when and only when K is lattice equivalent to one of the following three pentagons:*

```
. o .       . o .        o o .
o . o       o . o        . . o
o . o       o o .        o . o
```

```
Figure 2-1
Only convex lattice polygons with v=5
and smallest g
```

Definition. A lattice polygon is *lean* if all its boundary lattice points are vertices.

In other words, a lattice polygon, K, is lean if $b(K) = v(K)$.

Coleman's Lemma ([2]). *Let $ABCDE$ be a convex lattice pentagon. If $\triangle ACE$ has no interior lattice points, then AC or CE must contain an interior lattice point. If, furthermore, AE has at least one interior lattice point, then both AC and EC contain an interior lattice point.*

We will give a more combinatorial proof than Coleman's original proof in [2].

Proof. In pentagon $ABCDE$, assume that $\triangle ACE$ contains no interior lattice points. Let X be the lattice point inside $\triangle ABC$ that is closest to AC. If there is no such lattice point, take X be B. Similarly, let Y be the lattice point inside $\triangle CDE$ that is closest to CE. If there is no such lattice point, take Y to be D. (See figure 2-2a.)

Figure 2-2a and 2-2b

Pentagon $AXCYE$ contains an interior lattice point by the Lattice Pentagon Theorem. This lattice point cannot be inside triangle AXC by hypothesis. It cannot be inside triangle ABC for then it would be closer to AC than X is, contradicting the manner in which point X was chosen. Similarly, it cannot be inside triangle CYE. Therefore, it must lie on either AC or CE.

Now suppose that, in addition, edge AE contains a lattice point, P. We have already seen that either AC or CE contains a lattice point. Without loss of generality, assume that lattice point X lies on AC (figure 2-2b). Again, let Y be the lattice point inside $\triangle CDE$ that is closest to CE. If there is no such lattice point, let Y be D. Then $CXPEY$ is a convex lattice pentagon, so by the Lattice Pentagon Theorem, it must contain an interior lattice point. This lattice point cannot occur in $\triangle CYE$ or it would be closer to CE than Y and it cannot lie inside $\triangle ACE$ by hypothesis. Hence it must lie on CE. Thus both AC and CE contain lattice points. $\square$

The Fat Pentagon Theorem. *Let K be a convex lattice pentagon and suppose one edge of K contains two interior lattice points. Then K contains at least two interior lattice points.*

Proof. From Census-1, we see that if K had only one interior lattice point, no edge of K would contain two interior lattice points. $\square$

The following proposition follows from the Central Hexagon Theorem.

Proposition 2.4. *If $v = 6$ then $g \geq 1$. Equality occurs when and only when K is lattice equivalent to the centrally symmetric hexagon shown in figure 2-3.*

```
. o o
o . o
o o .
```

Figure 2-3
Unique convex lattice polygon with v=6
and smallest g

The Fat Hexagon Theorem. *A non-lean convex lattice hexagon contains at least two interior lattice points.*

Proof. If the hexagon had exactly one interior lattice point, then it would have to be lean by the Central Hexagon Theorem. □

Arkinstall [1] showed that $v = 7$ implies $g \geq 2$. We will give a slightly simpler proof and then show that $v = 7$ implies $g \geq 4$.

Lemma 2.5. *If $v = 7$ then $g \geq 2$.*

Figure 2-4

Proof. Let $ABCDEFG$ be a convex lattice heptagon. Then $ABCDE$ is a convex lattice pentagon so must contain an interior lattice point, X, by the Lattice Pentagon Theorem (see figure 2-4). Then $AXEFG$ is another convex lattice pentagon, so it must contain an interior lattice point, Y. □

Proposition 2.6. *If $v \geq 7$, then the interior lattice points of K can not colline.*

Proof. Suppose $v \geq 7$ and that all the lattice points interior to K lie on a line L. Since $v \geq 7$ implies $g \geq 4$, there are at least two lattice points, say P and Q, on line L inside K. Line L can meet K at at most 2 points, so there are at least 5 vertices of K that do not lie on L. Line L divides the plane into two regions, and we have 5 points, so at least 3 of these vertices, say A, B, and C, lie in one of the regions. Then $ABCPQ$ would be a convex lattice pentagon and thus we would have an interior lattice point by the Lattice Pentagon Theorem. The existence of this point contradicts the fact that all the lattice points interior to K lie on L. □

Lemma 2.7. *Let K be a convex lattice polygon. If $v = 7$ and $g \leq 3$ then the line joining any two interior lattice points must pass through two vertices of K.*

Proof. Let X and Y be any two interior lattice points. The line XY divides the heptagon into two pieces. If one of these pieces contains exactly 1 or 2 vertices (not on XY) and XY does not pass through 2 vertices, this would be a contradiction, for in the other piece, we would be able to create a heptagon ($XYCDEFG$ in figure 2-5a), thereby finding another 2 interior lattice points by lemma 2.5 or we would be able to find a non-lean hexagon ($XCDEFG$ in figure 2-5b) also implying 2 more interior lattice points by the Fat Hexagon Theorem.

`Figure 2-5a and 2-5b`

If one of these pieces contains exactly three vertices (not on XY) and XY doesn't pass through two vertices, this would also be a contradiction, for we would find two pentagons present, one in each piece ($XABCY$ and $YDEFX$ in figure 2-6), thereby finding another two interior lattice points by the Lattice Pentagon Theorem.

`Figure 2-6`

This covers all cases. $\square$

Proposition 2.8. *If $v = 7$ then $g \geq 4$. Equality can hold as can be seen by the heptagons in figure 2-7 in which $v = 7$ and $g = 4$.*

```
. o . .     . o o .     . o o .     . o o .
o . . o     . . . o     . . . o     o . . o
o . . o     o . . o     o . . o     . . . o
. o o .     o . o .     o o . .     o . o .
```

Figure 2-7
Some convex lattice polygons with v=7
and smallest g

Proof. Let $ABCDEFG$ be a convex lattice heptagon. There must be two interior lattice points, X and Y, by lemma 2.5. Then, by lemma 2.7, line XY passes through two vertices, say P and Q. Of the other 5 vertices of the heptagon, at least 3 of them must fall on one side of PQ. Call these A, B and C, in order, with A nearest to P.

Figure 2-8

Pentagon $XYCBA$ must contain a lattice point. Call it Z. Suppose $g = 3$. Then by lemma 2.7, XZ must pass through B or C. If it passes through C we would get pentagon $GABCX$ yielding another interior lattice point, a contradiction (see figure 2-8). Thus XZ must pass through B. In the same manner, we find that YZ must also pass through B. This is a contradiction because line BZ cannot pass through both X and Y. Thus, the assumption that $g = 3$ is false and we must have $g \geq 4$. $\square$

There are many figures for which equality holds, so we will not bother to list them all here. A few examples are shown above in figure 2-7.

The Central Octagon Theorem. *If K is a convex lattice polygon with $v = 8$ and $g = 4$, then K is lattice equivalent to the centrally symmetric octagon shown in figure 2-9.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 2-9
Unique convex lattice polygon with v=8
and smallest g

Proof. Let the octagon be $ABCDEFGH$. Quadrilateral $ABCD$ can't contain an interior lattice point, P, for then $APDEFGH$ would be a 7-gon and we would thus have an additional 4 lattice points interior to K (by Proposition 2.8). Therefore $AD \parallel BC$ by the Lattice Trapezium Theorem and Census-0. Similarly, $HC \parallel AB$.

Let diagonals AD and CH meet at point P. We have just shown that $ABCP$ is a parallelogram. Since points A, B, and C are lattice points, it follows that P must be a lattice point.

In a similar manner, we find the other three interior lattice points, Q, R, and S and see that the four interior lattice points form a parallelogram. A suitable integral unimodular affine transformation transforms this parallelogram into a square. This transformation also forces each vertex of the octagon to be in fixed positions on the extensions of the sides of the square; so we see that the resulting octagon is lattice equivalent to the one shown. □

Corollary 2.9. *If $v = 8$ then $g \geq 4$. Equality occurs when and only when K is lattice equivalent to the centrally symmetric octagon shown in figure 2-9.*

Proof. If $v = 8$, then remove one vertex to get a convex lattice polygon with $v = 7$ which implies there are at least 4 interior lattice points (by Proposition 2.8). The equality condition follows from the Central Octagon Theorem. □

The Quadrangular Segment Theorem. *Let A, B, C, and D be four consecutive vertices of a convex lattice polygon, K, with $v > 4$. Then either quadrilateral $ABCD$ contains an interior lattice point, or chord AD contains an interior lattice point.*

Proof. (following [2]) Let E be the vertex of K adjacent to D and on the other side from C. Since $ABCDE$ is a convex lattice pentagon, by Coleman's Lemma, there is either a lattice point inside $\triangle ACD$, or there is a lattice point on AC or on AD. □

Theorem 2.10. *Let PA, PB, PC be three distinct diagonals of a convex lattice polygon. Then quadrilateral $PABC$ contains an interior lattice point.*

Proof. (following [2]) Let X be a vertex of K on the side of PA that does not contain B. Let Y be a vertex of K on the side of PC that does not contain B. (See figure 2-10.)

Figure 2-10

Suppose that quadrilateral $PABC$ did not contain an interior lattice point. Applying Colemen's Lemma to pentagon $PABCY$, we find that diagonal PC must contain an interior lattice point, say Z. Applying Coleman's Lemma to pentagon $PXABC$, since PC contains an interior lattice point, we must either have a lattice point inside $\triangle PAC$ or a lattice point in the interior of segment AC. In either case, we reach a contradiction, having found a lattice point inside quadrilateral $PABC$. $\square$

Proposition 2.11. *If w_x is the horizontal width of a lean convex lattice polygon, K, then $v(K) \leq 2(w_x + 1)$.*

Recall that the horizontal width of a convex figure is the distance between its two vertical support lines.

Proof. The two vertical support lines must both be of the form $x = k$ where k is an integer. There are $w_x + 1$ vertical lattice lines between and including these two support lines. Each such line intersects K in at most two points, and every vertex of K must lie on at least one of these lines. Hence $v(K) \leq 2(w_x) + 1$. $\square$

The same result holds true for the vertical width (the distance between the two horizontal support lines). That is, if w_y is the vertical width of a lean convex lattice polygon, K, then $v(K) \leq 2(w_y + 1)$.

3. The Interior Hull

Definition. Let K be a convex body in the plane. Then $H(K)$ is the boundary of the convex hull of the lattice points interior to K. $H(K)$ is called the *interior hull* of K.

This will frequently be denoted by just H, if K is fixed.

Loosely speaking, H is the largest convex lattice polygon contained within K. Note, however, that H might degenerate into a line segment, a point, or the null set.

In this section, we will investigate the relationship between a convex lattice polygon, K, and its interior hull. In particular, we will show that the number of vertices of the interior hull must be at least half the number of vertices of K (if $v(K)$ is large enough). Also, we will show that the number of lattice points on the boundary of the interior hull must be at least 2/3 the number of vertices of K (if $v(K)$ is large enough).

Definition. Let K be a convex polygon with edge AB. Then $h(AB)$ denotes the open halfplane bounded by AB that is exterior to K.

Proposition 3.1 (The 3-vertex Restriction). *Let K be a convex lattice polygon and let H be the interior hull of K. Let AB be an edge of H. Then $h(AB)$ contains at most two vertices of K.*

Figure 3-1

Proof. Suppose this open halfplane contains 3 vertices of K, say X, Y, and Z (see figure 3-1). Consider the five points: A, B, X, Y, and Z. The point X can not be in the convex hull of the other four points because then X would be an interior point of K and not a vertex of K. A similar argument holds for Y and Z. Thus $ABXYZ$ is a convex lattice pentagon with no interior lattice points contradicting the Lattice Pentagon Theorem. $\square$

Theorem (The Interior Hull Vertex Inequality). *Let K be a convex lattice polygon and let $H = H(K)$. If $v(K) \geq 7$, then $v(K) \leq 2v(H)$.*

Proof. Since $v \geq 7$, the interior lattice points of K do not colline. Thus, the interior hull forms a polygon. The number of edges of this polygon is at most $v(H)$. By the lemma, for each edge AB of this polygon, $h(AB)$ contains at most two vertices of K. These halfplanes cover all of the vertices of K. Thus the total number of vertices of K is at most $2v(H)$. Hence $v(K) \leq 2v(H)$. $\square$

Corollary 3.2. *Let K be a convex lattice polygon with interior hull H. If $v(K) \geq 7$, then $v(H) \geq \lceil \frac{1}{2} v(K) \rceil$.*

Corollary 3.3. *Let K be a convex lattice polygon with $v \geq 9$. Then the interior lattice points of K do not lie on two parallel lines.*

Proof. If the interior lattice points fell on two parallel lines, then the boundary of the interior hull would have at most 4 vertices. Thus $v(H) \leq 4$ which contradicts the fact that (by the Interior Hull Vertex Inequality) $v(H) \geq v(K)/2 \geq 9/2 > 4$. $\square$

It is not known if the coefficient "2" in the Interior Hull Vertex Inequality is best possible for large v.

As for inequalities in the reverse direction, it might be thought that the interior hull could not have more vertices than the original polygon. However, this conjecture is false as can be seen by the following figure in which $v(H) > 2v(K)$. The plus signs represent the vertices of H.

```
o . . . . . . . .
. + + . . . . . .
. . . . . + . . .
. . . . . . . + o
. . . . . + . . .
. + + . . . . . .
o . . . . . . . .
```

Figure 3-2
Convex lattice polygon with v(H)>2v(K)

Theorem (The Interior Hull Boundary Inequality). *Let K be a convex lattice polygon and let $H = H(K)$. If $v(K) \geq 9$, then $2v(K) \leq 3b(H)$.*

Proof. By the Interior Hull Vertex Inequality, we see that $v(K) \geq 9$ implies $v(H) \geq 5$. By the Lattice Pentagon Theorem, we have $g(H) \geq 1$. Thus, there is a lattice point, P, in the interior of H.

If we draw rays from P to each of the lattice points on the boundary of H, there will be no lattice points between any two adjacent rays. Also, the angle between two adjacent rays will always be smaller than π. We have thus divided H into at most $b(H)$ wedges.

Figure 3-3

For purposes of this proof, define an *element* of K to be either a vertex of K or an (open) edge of K. Consider any wedge, with rays PX and PY where X and Y are successive lattice points along the boundary of H. The wedge consists of the two bounding rays and the space between them.

Claim. *No wedge intercepts 5 or more elements of K.*

The elements of K alternate: vertex, edge, vertex, $\ldots$.

A wedge cannot intercept the elements: (vertex, edge, vertex, edge, vertex) since that would contradict the 3-vertex Restriction (figure 3-4a).

A wedge cannot intercept the elements: (edge, vertex, edge, vertex, edge) for then there would be two vertices of K, A and B strictly inside the wedge (figure 3-4b). Let Q be the lattice point in the interior of H that is strictly inside the wedge and is closest to segment XY. If there are no such points, set Q to be P. Then $ABXQY$ is a convex lattice pentagon implying that the wedge contained another interior lattice point. This is a contradiction.

Figure 3-4a and 3-4b

Thus each wedge intercepts at most 4 elements of K. Since each ray intercepts exactly one element, these elements will be counted twice if we add up all the elements intercepted by the wedges. There are exactly $b(H)$ such elements. Thus the total number of elements in all can't be more than $4b(H)$ minus $b(H)$. But, the total number of elements is just $2v(K)$, so $2v(K) \leq 4b(H) - b(H)$ or $3b(H) \geq 2v(K)$. $\square$

Corollary. *Let K be a convex lattice polygon with interior hull H. If $v(K) \geq 9$ then $b(H) \geq \lceil \frac{2}{3} v(K) \rceil$.*

Proposition 3.4 (The Outer Parallel Condition). *Let K be a convex lattice polygon with interior hull H. Let XY be an edge of H. If $h(XY)$ contains two vertices of K, say A and B, then $AB \parallel XY$.*

Proof. First note that by the 3-vertex Restriction (Proposition 3.1), $h(XY)$ contains at most 2 vertices of K. Since $ABXY$ is a quadrilateral containing no interior lattice points, it must be a trapezoid by the Lattice Trapezium Theorem. If AB is not parallel to XY, then we would have $AY \parallel BX$. But then, from Census-0, we would find that there would be a lattice point on either AY or BX, a contradiction. Thus $AB \parallel XY$. $\square$

Note that in fact, AB is the parallel lattice line closest to XY. (A *lattice line* is a line through two lattice points.) This gives us an aid in locating the vertices of K if we are given H. For each edge of H, we draw the parallel lattice line outside H that is closest to that edge. The vertices of K must then lie on these "outer parallel" lines. Each of these lines will contain 0, 1, or 2 vertices of K.

4. Properties of $g(v)$

Notation. Let $g(v) = \min\{g(K) | v(K) = v\}$ where the minimum is taken over all convex lattice polygons, K.

We have already shown that $g(3) = 0$, $g(4) = 0$, $g(5) = 1$, $g(6) = 1$, $g(7) = 4$, and $g(8) = 4$. We wish now to study the properties of $g(v)$.

Note that there should be no confusion between this function, $g(v)$ and the lattice point counting function, $g(K)$, since the domain of $g(v)$ is the set of positive integers, whereas the domain of $g(K)$ is the set of convex lattice polygons in the plane.

The earliest bound on $g(v)$ comes from Scott's Bound for b: $b \le 2g + 6$ (for $g > 0$ and $v > 3$). Since $v \le b$, this gives us the inequality $g \ge (v - 6)/2$, so we have

$$g(v) \ge \left\lceil \frac{v}{2} \right\rceil - 3.$$

This bound is very crude. In this section of this paper, we will find better bounds for $g(v)$.

Proposition 4.1. *The function $g(v)$ is monotone.*

Proof. Let K be any convex v-gon. Remove one vertex from K to get a convex $(v-1)$-gon called K^*. Polygon K^* has at least $g(v-1)$ interior lattice points. Since each K has at least $g(v-1)$ interior lattice points, so must the the minimum over all K have at least $g(v-1)$ interior lattice points. Thus $g(v) \ge g(v-1)$. $\square$

Lemma 4.2. *If $v \ge 5$, then $g(v+2) \ge g(v) + 1$.*

Figure 4-1

Proof. Let $A_1A_2A_3A_4A_5 \ldots A_{v+2}$ be a convex $(v{+}2)$-gon with $v \geq 5$. Polygon $A_1A_2A_3A_4A_5$ is a convex lattice pentagon, so it must contain a lattice point, P, in its interior (see figure 4-1). Polygon $A_1PA_5A_6 \ldots A_{v+2}$ is a convex lattice v-gon, so it must contain $g(v)$ additional lattice points. Thus K contains at least $g(v)+1$ interior lattice points. □

Proposition 4.3. *If $v \geq 5$, then $g(v+2) \geq g(v)+2$.*

Proof. Let K be a convex $(v+2)$-gon and let H be the interior hull of K. By proposition 2.8, K contains at least 4 interior lattice points. Let P and Q be two lattice points on the boundary of H.

Line PQ divides K into 2 parts. Let $h(PQ)$ be the open halfplane bounded by PQ that contains no points of H. Let $h^*(PQ)$ be the other open halfplane bounded by PQ.

Halfplane $h(PQ)$ must contain fewer than 3 vertices of K by the 3-vertex Restriction. Line PQ can contain at most 2 vertices of K and $h(PQ)$ can contain at most 2 vertices of K, so $h^*(PQ)$ contains at least $v-2$ vertices of K (figure 4-2).

`Figure 4-2`

These plus P and Q yield a convex v-gon which must have at least $g(v)$ interior lattice points by definition. Thus K contains at least $g(v)+2$ interior lattice points. □

Corollary 4.4. *If $v = 9$ then $g \geq 6$.*

This follows immediately from the fact that $g(7) = 4$.

Proposition 4.5. *If $n \geq 4$ then $g(2n-1) \geq 2n-4$ and $g(2n) \geq 2n-4$.*

This follows by induction on n and the fact that $g(7) = g(8) = 4$.

Corollary 4.6. *If $v \geq 7$, then $g(v) \geq 2\lfloor \frac{v-3}{2} \rfloor$.*

This improves the inequality $g(v) \geq \lfloor \frac{v-2}{2} \rfloor$ (for $v \geq 7$) found by Coleman [2].

Corollary 4.7. *If $v \geq 7$, then $g(v) \geq v-4$.*

This improves the inequality $g(v) \geq v-5$ (for $v \geq 7$) found by Coleman [2]. In Coleman's paper, Coleman also made a conjecture that is related to the present topic. It is a generalization of Scott's Bound for b.

Coleman's Conjecture. *If a convex lattice polygon has v vertices, g interior lattice points ($g > 0$), and b boundary lattice points, then $b \leq 2g + 10 - v$.*

The conjecture is known to be true for $v = 3$ and $v = 4$ by Scott's Bound for b. Rabinowitz ([3], theorem 5.1.4) has shown it to be true for $v = 5$. The conjecture is still unproven for arbitrary v.

In Corollary 4.7, we have shown (for sufficiently large v) that $g(v)$ is at least v (minus a constant). We will now show (for sufficiently large v) that $g(v)$ is at least $\frac{3}{2}v$ (minus a constant). First we start with a lemma.

Lemma 4.8. *Let K be a convex lattice polygon with v vertices and g interior lattice points. Let H be the interior hull of K. If $g(K) = g(v-2) + 2$, then H is lean.*

Proof. If H were not lean, then there would be some support line of H, L, containing 3 or more lattice points on the boundary of H, say P, Q, and R. L divides K into two pieces. Let K_1 be the piece that does not contain any portion of H in its interior and let K_2 be the other piece. Then K_1 must have fewer than 3 vertices (not counting any on L) by the 3-vertex Restriction.

Thus K_2 must contain at least $v - 2$ vertices (including any that might be endpoints of L). But any $(v-2)$-gon must contain at least $g(v-2)$ interior lattice points. Those, plus the 3 on L show that K contains at least $3 + g(v-2)$ lattice points, a contradiction. Thus H is lean. □

In general, it is not true that an edge of H must pass through any vertices of K; for example, see figure 4-3. However, we shall show that if g is small enough, this condition must obtain.

```
. . o . o . .
. . + . + . .
o + . . . + o
. . . . . . .
o + . . . + o
. . + . + . .
. . o . o . .
```

Figure 4-3
Convex lattice polygon with no edge of H
passing through a vertex of K

Proposition 4.9. *Let K be a convex v-gon with at most $g(v) + 1$ interior lattice points. Then each edge of $H(K)$ passes through some vertex of K.*

Proof. Let XY be an edge of H. Then $h(XY)$ contains at most 2 vertices of K. Let $h^*(XY)$ denote the vertices of K not in $h(XY)$. If XY passes through no vertices of K, then $h^*(XY)$ contains at least $v - 2$ vertices of K. These plus X and Y form a convex v-gon. It has at least $g(v)$ interior lattice points. Thus K has $g(v) + 2$ interior lattice points, a contradiction. □

Theorem 4.10. *If $v \geq 2n+1$ then $g \geq 3n-5$.*

Proof. We proceed by induction. The theorem has already been shown to be true if $n = 2$ or $n = 3$. (It is trivially true for $n = 0$ and $n = 1$.) So suppose it is true for all integers smaller than n, we will now show it is true for n ($n \geq 4$).

Let K be a convex lattice polygon with $2n+1$ sides with $n > 1$. By the induction hypothesis, we know that $g(2n-1) \geq 3(n-1) - 5 = 3n - 8$. Hence, by Proposition 4.3, $g(2n+1) \geq 3n - 6$.

Let H be the interior hull of K. If $G(H) = 3n-5$, then we are done, and since $G(H) = g(2n+1) \geq 3n-6$, we may assume that $G(H) = 3n-6$.

By Lemma 4.8, H is lean. Thus $v(H) = b(H)$.

Claim. $v(H) < \lfloor \frac{4n+4}{3} \rfloor$.

Proof. Suppose H has $\lfloor \frac{4n+4}{3} \rfloor$ or more vertices. There are three cases to consider, depending on the remainder when n is divided by 3. We will reach a contradiction by showing that in each case, $G(H) > 3n-6$.

We first note that $v(H) > n+1$. This is because $\lfloor \frac{4n+4}{3} \rfloor \geq \frac{4n+4}{3} - 1 = \frac{4n+1}{3} > n+1$ (since $n > 2$).

Case 1: $n = 3k$.

In this case, $v(H)$ is at least $\lfloor \frac{12k+4}{3} \rfloor = 4k+1$. But $4k+1 < 2n+1$, so by the inductive hypothesis, $g(H) \geq 6k-5 = 2n-5$. Thus $G(H) \geq v(H) + g(H) > (n+1) + (2n-5) = 3n-4 > 3n-6$, the desired contradiction.

Case 2: $n = 3k+1$.

In this case, $v(H)$ is at least $\lfloor \frac{12k+8}{3} \rfloor = 4k+2 > 4k+1$. But $4k+1 < 2n+1$, so by the inductive hypothesis, $g(H) \geq 6k-5 = 2n-7$. Thus $G(H) \geq v(H) + g(H) > (n+1) + (2n-7) = 3n-6$, the desired contradiction.

Case 3: $n = 3k+2$.

In this case, $v(H)$ is at least $\lfloor \frac{12k+12}{3} \rfloor = 4k+4 > 4k+3$. But $4k+3 < 2n+1$, so by the inductive hypothesis, $g(H) \geq 6k-2 = 2n-6$. Thus $G(H) \geq v(H) + g(H) > (n+1) + (2n-6) = 3n-5 > 3n-6$, the desired contradiction.

This proves our claim. $\square$

We have just shown that $v(H) = b(H) < \lfloor \frac{4n+4}{3} \rfloor$ or $b(H) \leq \lfloor \frac{4n+4}{3} \rfloor - 1$. By the Interior Hull Boundary Inequality, we have

$$v(K) \leq \frac{3}{2} b(H) \leq \frac{3}{2} \left\lfloor \frac{4n+4}{3} \right\rfloor - \frac{3}{2} \leq \frac{3}{2} \left(\frac{4n+4}{3} \right) - \frac{3}{2} = 2n + \frac{1}{2} < 2n+1$$

contradicting the fact that $v(K) = 2n+1$. Hence our assumption that $g(K) = 3n-6$ is incorrect, and we must have $g(K) \geq 3n-5$. $\square$

Corollary 4.11. $g(2n+1) \geq 3n-5$.

Corollary 4.12. $g(2n+2) \geq 3n-5$.

Corollary 4.13. $g(v) \geq 3\lfloor \frac{v-1}{2} \rfloor - 5$.

This comes from combining the previous two inequalities. Also note that the result is trivially true if $v = 3$ or $v = 4$.

Corollary 4.14. $g(v) \geq \frac{3}{2}v - 8$.

We now prove a few properties about convex lattice polygons, K, with $g(K) = g(v)$.

Lemma 4.15. *Let $ABCDE\ldots$ be a convex lattice v-gon, K, with $g(K) = g(v)$. Then triangle ABC has no interior lattice points.*

Proof. If there were a lattice point, X, in the interior of $\triangle ABC$, then $AXCDE\ldots$ would be a convex lattice v-gon so would have at least $g(v)$ interior lattice points. This would be a contradiction because it would show that $g(K)$ was at least $g(v) + 1$. $\square$

Proposition 4.16. *For any integer v ($v \geq 3$), there is a lean convex lattice v-gon, K, with $g(K) = g(v)$.*

Proof. By the definition of $g(v)$, there is at least one convex lattice v-gon, K, with $g(K) = g(v)$. Let $ABCDE\ldots$ be a convex lattice v-gon, K, with $g(K) = g(v)$ and minimal b. Suppose K were not lean. Then there would be a lattice point, X, in the interior of some edge, say AB. Triangle ABC contains no interior lattice points by Lemma 4.15. Thus $AXCDE\ldots$ would be a convex lattice v-gon witht he same number of interior lattice points as K, but with smaller b. This is a contradiction. $\square$

We now move on to considering the cases $v = 9$ and $v = 10$.

Proposition 4.17. *If $v = 9$ then $g \geq 7$. Equality holds when and only when K is lattice equivalent to the nonagon shown in figure 4-4.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

Figure 4-4
Unique convex lattice polygon with v=9
and smallest g

Proof. Letting $v = 9$ in Corollary 4.13 shows that $g \geq 7$.

To show that the pictured polygon is unique, we can proceed as follows. Let H be the convex hull of a lattice nonagon, K, with $g(K) = 7$. Since $v(K) = 9$, we must have $b(H) \geq 6$ and $v(H) \geq 5$ by the Interior Hull Inequalities. Since $v(H) \leq b(H) \leq G(H) = g(K) = 7$, $v(H)$ is either 5, 6, or 7. We can't have $v(H) = 7$, for a lattice heptagon would have another 4 interior lattice points, making $g(K)$ at least 11. If $v(H) = 5$, then $g(H) \geq 1$, so $b(H) \leq 6$. Thus $b(H) = 6$ and H is not lean. By Census-1, we see that H must be lattice equivalent to the pentagon ABCDE shown in figure 4-5a.

```
. . c . .          . . o o o
. b A d .          . o + + o
a E . B e          o + . + o
n D . C f          o + + o .
m k j h g          o o o . .
```

Figures 4-5a and 4-5b

By the Outer Parallel Condition (Proposition 3.4), the vertices of K must be 9 out of the 12 lattice points marked a-n in figure 4-5a. Clearly, point c must be a vertex of K since bd cannot be an edge of K. Since each of the 5 outer parallel lines can contain at most 2 vertices of K, the edges of K must cut away the four vertices e, g, m, and a. But this leaves only 8 lattice points left, contradicting the fact that K is a 9-gon.

Thus H must be a hexagon. Also, H must be lean because otherwise there would be 7 lattice points on the boundary of H and at least one inside, contradicting the fact that $g(K) = 7$. Thus H is a lattice hexagon and $g(H) = 1$ so H is uniquely determined by the Central Hexagon Theorem. We show H in figure 4-5b as the six points marked by plus signs. The Outer Parallel Condition limits the vertices of K to be 7 of the 12 lattice points marked by circles. Symmetry considerations then shows that K must be lattice equivalent to figure 4-4. $\square$

Corollary 4.18. *A non-lean convex lattice nongaon must contain at least 8 interior lattice points.*

Lemma 4.19. *If $v = 10$, then $g \geq 9$.*

Proof. Let H be the interior hull of K, a lattice decagon. Then $b(H) \geq 7$ and $v(H) \geq 5$ by the Interior Hull Inequalities. If $v(H) \geq 7$, then $g(H) \geq 4$, so $g(K) \geq 11$ and we would be done. Hence we may assume that $v(H) < 7$. In that case, H cannot be lean, since $b(H) \geq 7$. Thus some edge of H, say XY, contains an interior lattice point Z. (See figure 4-5.)

Figure 4-5

Without loss of generality, we may assume that K is a decagon with the smallest number of interior lattice points; i.e. $g(K) = g(10)$. Then by Proposition 4.9, edge XY must pass through at least one vertex (say A) of the decagon $ABCDEFGHIJ$. By the

3-vertex Restriction, $h(XY)$ cannot contain 3 vertices of K, so $h(XY)$ cannot contain vertex D (although XY might pass through D). Thus $AYEFGHIJ$ is a convex lattice octagon. It must contain at least 6 interior lattice points by the Fat Octagon Theorem. Hence K must have at least 9 interior lattice points (these plus X, Y, and Z). $\square$

It should also be noted that if some edge of H does not pass through two vertices of K, then K must contain at least 10 interior lattice points. For, as in the above proof, assume AXY does not pass through D. Then $AYDEFGHIJ$ would be a non-lean convex lattice nonagon and would thus contain at least 8 interior lattice points by Corollary 4.18. These plus X and Y would show that K contained at least 10 interior lattice points.

Lemma 4.19 shows that $g(10)$ is at least 9. In an attempt to determine the precise value of $g(10)$, the author wrote a computer program that generated all convex lattice polygons within the rectangle bounded by $x = -10$, $x = 10$, $y = 0$, and $y = 10$. Examining this collection of lattice polygons, it was found that when $v = 10$, g was always at least 10. Furthermore, there was precisely one convex lattice decagon with $g = 10$ (it is shown below in figure 4-6). This is strong evidence for the following conjecture; however, there is no proof that some decagon with $g = 9$ might have been missed by the computer search (because it does not fit in the rectangle limiting the search).

Conjecture 4.20. *If $v = 10$, then $g \geq 10$. Equality holds when and only when K is lattice equivalent to the decagon shown in figure 4-6.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

Figure 4-6
Conjectured unique convex lattice polygon with v=10
and smallest g

We note that this polygon is lean. Hence we have as an immediate corollary:

Conjecture (The Fat Decagon Theorem). *A non-lean convex lattice decagon contains at least 11 interior lattice points.*

The following conjecture is suggested by figures 2-3, 2-9, and 4-6:
Conjecture 4.21. If v is even, $v > 4$, and K is a convex lattice v-gon with $g(K) = g(v)$, then K is central symmetric.

Lemma 4.22. *If $v = 11$, then $g \geq 11$.*

Proof. We will follow the same method of proof as used to prove Lemma 4.19. Let K be a convex lattice 11-gon. Let H be the interior hull of K. By the Interior Hull Inequalities, we have $v(H) \geq 6$ and $b(H) \geq 8$. If $v(H) \geq 7$, then we would be done since a 7-gon must contain at least 4 interior lattice points. Thus we may assume that $v(H) = 6$. But since $b(H) \geq 8$, we conclude that H is not lean. Thus H contains some edge XY that contains an interior lattice point, Z. By the 3-vertex Restriction, $h(XY)$ contains at most 2 vertices of K. Since XY passes through at most 2 vertices of K, this means that there are at least 7 vertices of K in $h^*(XY)$, the open halfplane bounded by XY on the same side as H. These 7 vertices plus X and Y form a convex lattice 9-gon, P. Since P is not lean, it must contain at least 8 interior lattice points by Corollary 4.18. These 8 lattice points plus X, Y, and Z, show that K contains at least 11 interior lattice points. $\square$

In Corollary 4.14, we showed that $g(v) \geq \frac{3}{2}v - 8$. We now improve this (for sufficiently large v) to $g(v) \geq \frac{3}{2}v - 6$.

Proposition 4.23. *If $v \geq 10$, then $g(v) \geq \lceil \frac{3}{2}v \rceil - 6$.*

Proof. We will show that for $v \geq 10$, $g(v) \geq \frac{3}{2}v - 6$. The result then follows since $g(v)$ must be an integer. We will proceed by induction on v. The proposition is already known to be true for $v = 10$ and $v = 11$, so assume $v \geq 12$. Let H be the interior hull of K, a convex lattice v-gon. By the Interior Hull Inequalities, we know that $v(H) \geq v/2$ and $b(H) \geq 2v/3$.
Case 1: $v(H) \geq \frac{2}{3}v$. In this case, by Corollary 4.14, we have $g(H) \geq \frac{3}{2}\left(\frac{2}{3}v\right) - 8 = v - 8$. Thus $g(K) = g(H) + b(H) \geq v - 8 + \frac{2}{3}v$. But, for $v \geq 12$, this is greater than or equal to $\frac{3}{2}v - 6$ and we are done.
Case 2: $v(H) < \frac{2}{3}v$. In this case, H is not lean because $b(H) \geq \frac{2}{3}v$. Thus there is some edge of H, XY, that contains an inerior lattice point, Z. Halfplane $h(XY)$ contains at most 2 vertices of K and XY passes through at most two vertices of K, so the open halfplane bounded by XY on the same side as H contains at least $v - 4$ vertices of K. These plus X and Y form a convex lattice $(v-2)$-gon, P. By the induction hypothesis, P contains at least $\frac{3}{2}(v-2) - 6$ interior lattice points. These plus X, Y, and Z, show that K contains at least $\frac{3}{2}v - 6$ interior lattice points. $\square$

Corollary 4.24. $g(12) \geq 12$.

Proposition 4.25. *$g(v) \geq v$ for $v \geq 11$.*

This follows from Proposition 4.3 and the fact that $g(11) \geq 11$ and $g(12) \geq 12$.

To find an upper bound for $g(v)$, we need only exhibit a polygon with v vertices and g interior lattice points.

Proposition 4.26. *There is a convex lattice polygon with $v = 2n$ and $g = \binom{n}{3}$.*

Proof. Let $A_1 = (0,0)$ and $B_1 = (1,0)$. We define A_k recursively by $A_{k+1} = A_k + (k+1, 1)$ for $k = 1, 2, \ldots, n-1$. That is, to get to A_{k+1} from B_k, you move right $k+1$ units and then up 1 unit. We define B_k recursively by saying that $B_{k+1} = B_k + (n+1-k, 1)$ for $k = 1, 2, \ldots, n-1$.

This polygon is shown in figure 4-7 for the case $n = 5$.

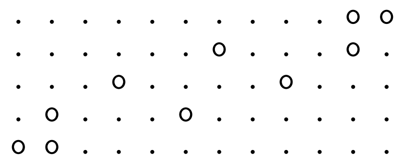

Figure 4-7

This polygon has $2n$ vertices, all at lattice points.

Since the abscissae increase in steps of $1, 2, \ldots, n-1$ for both the A_k and the B_k, it follows that A_n is one unit to the left of B_n since A_1 was one unit to the left of B_1. This fact, plus the way the slopes of the sides were chosen, assures us that the polygon is convex.

We will now count the number of lattice points interior to this polygon. The polygon has a height of $n-1$, so there are $n-2$ horizontal lines upon which interior lattice points may lie. They lie on the line segments $A_kB_k, k = 2, 3, \ldots n-1$. It is easy to sum up the abscissae to find

$$A_k = (\sum_{i=1}^{k-1} i, k-1)$$

and

$$B_k = (1 + \sum_{i=1}^{k-1} (n-i), k-1)$$

so that the distance from A_k to B_k is

$$\begin{aligned} 1 + \sum_{i=1}^{k-1}(n-i) - \sum_{i=1}^{k-1} i &= 1 + \sum_{i=1}^{k-1}(n-2i) \\ &= 1 + \sum_{i=1}^{k-1} n - 2\sum_{i=1}^{k-1} i \\ &= 1 + n(k-1) - k(k-1) = 1 + (n-k)(k-1). \end{aligned}$$

Thus the number of lattice points on this line segment and inside K is just $(n-k)(k-1)$. The total number of lattice points inside K is therefore

$$\begin{aligned} \sum_{k=1}^{n-1} (n-k)(k-1) &= \sum_{k=1}^{n-1} (n+1)k - \sum_{k=1}^{n-1} k^2 - \sum_{k=1}^{n-1} n \\ &= (n+1)\frac{n(n-1)}{2} - \frac{(n-1)n(2n-1)}{6} - (n-1)n \\ &= \frac{n(n-1)(n-2)}{6}. \end{aligned}$$

(We could start summing at $k = 1$ because we know that A_1B_1 contributes 0 to the sum.) This final answer shows that $g = \binom{n}{3}$ as claimed. $\square$

Since vertex A_1 can be removed from the polygon exhibited above without changing the number of interior lattice points, we have the following result:

Corollary 4.27. *There is a convex lattice polygon with* $v = 2n-1$ *and* $g = \binom{n}{3}$.

Corollary 4.28. $g(2n) \le n(n-1)(n-2)/6$ *and* $g(2n-1) \le n(n-1)(n-2)/6$.

Corollary 4.29. $g(n) \le \binom{\lceil n/2 \rceil}{3}$.

Corollary 4.30. $g(10) \le 10$, $g(11) \le 20$, $g(12) \le 20$, $g(13) \le 35$, $g(14) \le 35$, $g(15) \le 56$, $g(16) \le 56$, *and* $g(17) \le 84$.

Proposition 4.31. $g(11) \le 17$, $g(12) \le 19$, $g(13) \le 27$, $g(14) \le 34$, $g(15) \le 48$, *and* $g(16) \le 56$.

We need only exhibit the appropriate polygon. (See figures 4-8 through 4-13.)

```
. . . . . . o o .
. . . . o . . . o
. . . . . . . . o
. o . . . . . o .
o . . . . o . . .
o o . . . . . . .
```

Figure 4-8
Convex lattice polygon with v=11 and g=17

```
. . . . o o .
. . o . . . o
. o . . . . o
. . . . . . .
o . . . . o .
o . . . o . .
. o o . . . .
```

Figure 4-9
Convex lattice polygon with v=12 and g=19

```
. . . . . . . . . o o .
. . . . . . o . . . . o
. . . . o . . . . . . o
. . . . . . . . . . o .
. o . . . . . . o . . .
o . . . . o . . . . . .
o o . . . . . . . . . .
```

Figure 4-10
Convex lattice polygon with v=13 and g=27

```
. . . . . . . o o .
. . . . o . . . . o
. . o . . . . . . o
. o . . . . . . . .
. . . . . . . . o .
o . . . . . . o . .
o . . . . o . . . .
. o o . . . . . . .
```

Figure 4-11
Convex lattice polygon with v=14 and g=34

```
. . . . . . . . . . . . . o . o .
. . . . . . . . . o . . . . . . o
. . . . . . o . . . . . . . . . o
. . . . o . . . . . . . . . . o .
. . . . . . . . . . . . . o . . .
. o . . . . . . . . o . . . . . .
o . . . . . o . . . . . . . . . .
o o . . . . . . . . . . . . . . .
```

Figure 4-12
Convex lattice polygon with v=15 and g=48

```
. . . . . . . . . . . o o .
. . . . . . . o . . . . . o
. . . . o . . . . . . . . o
. . o . . . . . . . . . . .
. o . . . . . . . . . . o .
. . . . . . . . . . . o . .
o . . . . . . . . o . . . .
o . . . . . o . . . . . . .
. o o . . . . . . . . . . .
```

Figure 4-13
Convex lattice polygon with v=16 and g=56

We may summarize as follows:

Theorem 4.32. *Let $g(v) = \inf\{g(K) | v(K) = v\}$. Then*

a. $g(3) = 0$.
b. $g(4) = 0$.
c. $g(5) = 1$.
d. $g(6) = 1$.
e. $g(7) = 4$.
f. $g(8) = 4$.
g. $g(9) = 7$.
h. $g(10) =$ *9 or 10.*

i. $11 \leq g(11) \leq 17$.
j. $12 \leq g(12) \leq 19$.
k. $14 \leq g(13) \leq 27$.
l. $15 \leq g(14) \leq 34$.
m. $17 \leq g(15) \leq 48$.
n. $18 \leq g(16) \leq 56$.
o. $g(v) \geq g(v-2) + 2$ *for* $v \geq 7$.
p. $g(2n+1) \geq 3n - 5$.
q. $3\lfloor \frac{v-1}{2} \rfloor - 5 \leq g(v) \leq \binom{\lceil v/2 \rceil}{3}$.

It remains an open problem to find better bounds for $g(v)$ or a good asymptotic formula for $g(v)$. The author feels that the lower bound of $3\lfloor \frac{v-1}{2} \rfloor - 5$ is far from best-possible and makes the following conjecture:

Conjecture. $g(v) = O(v^3)$.

5. Inequalities with g fixed

We now look at the related problem of finding the bounds on v for any given value of g.

Proposition 5.1. *Let K be a convex lattice polygon. Then*

$$\inf\{v(K) | g(K) = g\} = 3.$$

Proof. We need only exhibit a triangle containing g interior lattice points for any given g. The triangle with vertices $(0,0)$, $(1,2)$, and $(g+1,1)$ has this property. □

A more interesting problem is to find the maximum value that v can have when we fix g.

Proposition 5.2. *If $g = 0$ then $v \leq 4$. Equality occurs when and only when K is lattice equivalent to the trapezoid whose vertices are $(0,0)$, $(p,0)$, $(0,1)$, and $(q,1)$ for some positive integers p and q.*

If v were greater than 4, there would be an interior lattice point by the Lattice Pentagon Theorem. The equality condition follows from Census-0.

Proposition 5.3. *If $g = 1$ then $v \leq 6$. Equality occurs when and only when K is lattice equivalent to the centrally symmetric hexagon shown in figure 5-1.*

```
. o o
o . o
o o .
```

Figure 5-1
Unique convex lattice polygon with g=1
and largest v

This follows from the fact that $v \geq 7$ implies $g \geq 4 > 1$. Equality is determined by the Central Hexagon Theorem.

Proposition 5.4. *If $g = 2$ then $v \leq 6$. Equality can hold as can be seen by figure 5-2 in which $v = 6$ and $g = 2$.*

```
. o o .      . o o .      . o . o      . o . o      . . o o
o . . o      o . . o      o . . o      o . . o      o . . o
. o o .      o . o .      o . o .      o o . .      o o . .
```

```
Figure 5-2
Some convex lattice polygons with g=2
and largest v
```

This follows from the fact that $v \geq 7$ implies $g \geq 4 > 2$.

Proposition 5.5. *If $g = 3$ then $v \leq 6$. Equality can hold as can be seen by figure 5-3 in which $v = 6$ and $g = 3$.*

```
. . o .
. . . o      . o o . .      . o . o .
o . . o      o . . . o      o . . . o
o . o .      . o o . .      o . . o .
```

```
Figure 5-3
Some convex lattice polygons with g=3
and largest v
```

This follows from the fact that $v \geq 7$ implies $g \geq 4 > 3$.

Proposition 5.6. *If $g = 4$ then $v \leq 8$. Equality holds when and only when K is lattice equivalent to the centrally symmetric octagon shown in figure 5-4.*

```
. o o .
o . . o
o . . o
. o o .
```

```
Figure 5-4
Unique convex lattice polygon with g=4
and largest v
```

This follows from the fact that $v \geq 9$ implies $g \geq 7 > 4$. Equality is determined by the Central Octagon Theorem.

Proposition 5.7. *If $g = 5$ then $v \leq 7$. Equality can hold as can be seen by figure 5-5 in which $v = 7$ and $g = 5$.*

```
. o o . .        . o o . .
o . . o .        o . . . .
o . . . .        o . . . o
. o . . o        . o . o .
```

Figure 5-5
Some convex lattice polygons with g=5
and largest v

We know that $v = 9$ implies that $g \geq 7$. Thus, if $g = 5$, then $v \leq 8$. However, it is not possible for g to be 5 and v to be 8 (we will show this below). Thus $g = 5$ implies $v \leq 7$.

This result is unusual enough to warrant calling it to the reader's attention.

Proposition (The Octagon Anomaly). *A convex lattice octagon can have 4 interior lattice points or 6 interior lattice points, but it can't have exactly 5 interior lattice points.*

This anomaly was first observed by Rabinowitz [3] in 1986, and proved by Steinberg [6] in 1988. The following proof is a simplification of Steinberg's proof.

Proof. Suppose we had a convex lattice octagon, K, that contained precisely 5 interior lattice points, A, B, C, D, and E. Let H denote the convex hull of these 5 points. By the Interior Hull Vertex Inequality, $v(H) \geq 4$. We cannot have $v(H) = 5$ by the Lattice Pentagon Theorem. Thus, $v(H) = 4$ and region H forms a quadrilateral, say $ABCD$. Lattice point, E, can lie inside this quadrilateral or on one of the edges (say CD).

Figure 5-6

Of the four halfplanes, $h(AB)$, $h(BC)$, $h(CD)$, $h(DA)$, each one can contain at most two of the vertices of K. Since K has exactly 8 vertices, we conclude that each halfplane contains exactly two vertices of K and the intersections of two halfplanes contain no vertices of K. Let X and Y be the two vertices of K in $h(AB)$ but not in $h(DA)$ or $h(BC)$ (see figure 5-6). (Points X and Y might lie on CB or DA extended.) Then $AEBXY$ is a convex lattice pentagon containing no interior lattice points, contradicting the Lattice Pentagon Theorem.

Thus our assumption that the lattice octagon K has exactly 5 interior lattice points has been proven to be incorrect. □

The Fat Octagon Theorem. *A non-lean convex lattice octagon contains at least 6 interior lattice points.*

Proof. The octagon must contain at least 5 interior lattice points by Corollary 2.9. It cannot contain exactly 5 interior lattice points by the Octagon Anomaly. Hence it must contain at least 6 interior lattice points. □

Proposition 5.8. *If $g = 6$ then $v \leq 8$. Equality can hold as can be seen by figure 5-7 in which $v = 8$ and $g = 6$.*

```
. o o . .
o . . . o
o . . . o
. o o . .
```

```
Figure 5-7
Convex lattice polygon with g=6
and largest v
```

This follows from the fact that $v \geq 9$ implies $g \geq 7 > 6$.

Proposition 5.9. *If $g = 7$ then $v \leq 9$. Equality occurs when and only when K is lattice equivalent to the nonagon shown in figure 5-8.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

```
Figure 5-8
Unique convex lattice polygon with g=7
and largest v
```

This follows from the fact that $v \geq 10$ implies $g \geq 10 > 7$. Equality follows from Proposition 4.28.

A computer search revealed the following interesting anomaly:

Conjecture (The Nonagon Anomaly). *A convex lattice nonagon can have 7 interior lattice points or 10 interior lattice points, but it can't have either 8 or 9 interior lattice points.*

The fact that 10 interior lattice points can occur is shown in figure 5-9.

```
. . o . .
. o . o .
. . . . .
o . . . o
o . . . o
. o . o .
```

Figure 5-9
A convex lattice nonagon with g=10

A consequence of this anomaly is the following two results:

Conjecture 5.10. *If $g = 8$ then $v \leq 8$. Equality can hold as can be seen by figure 5-10 in which $v = 8$ and $g = 8$.*

```
. o o . . .
o . . . . o
o . . . . o
. o o . . .
```

Figure 5-10
Convex lattice polygon with g=8
and largest v

If $g = 8$, then we must have $v \leq 9$ since $v = 10$ implies $g \geq 9$. However, assuming the Nonagon Anomaly is true, we can't have $v = 9$. Hence $v \leq 8$.

Conjecture 5.11. *If $g = 9$ then $v \leq 8$. Equality can hold as can be seen by figure 5-11 in which $v = 8$ and $g = 9$.*

```
. o o . . . .
o . . . . . o
o . . . . o .
. o o . . . .
```

Figure 5-11
Convex lattice polygon with g=9
and largest v

If $g = 9$, then we would have $v \leq 9$ since $v = 10$ implies $g \geq 10$ (by Conjecture 4.20). However, by the Nonagon Anomaly, we can't have $v = 9$. Hence $v \leq 8$.

Conjecture (The Fat Nonagon Theorem). *A non-lean convex lattice nonagon contains at least 10 interior lattice points.*

If $v = 9$, then by Proposition 4.17 we must have $g \geq 7$. But there is a unique polygon with $v = 9$ and $g = 7$ and it is lean. Hence we must have $g \geq 8$. Assuming the Nonagon Anomaly is true, $g = 8$ and $g = 9$ are ruled out; so we must have $g \geq 10$.

Proposition 5.12. *If $g = 10$ then $v \leq 10$.*

This follows from Lemma 4.22, for if v were greater than 10 then we would have $g \geq 11$.

Conjecture 5.13. *Equality in Proposition 5.12 occurs when and only when K is lattice equivalent to the decagon shown in figure 5-12.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

Figure 5-12
Conjectured unique convex lattice polygon with g=10
and largest v

This follows from conjecture 4.20.

We may summarize as follows:

Theorem 5.14. *Let $v(g) = \sup\{v(K)|g(K) = g\}$. Then*

a. $v(0) = 4$.
b. $v(1) = 6$.
c. $v(2) = 6$.
d. $v(3) = 6$.
e. $v(4) = 8$.
f. $v(5) = 7$.
g. $v(6) = 8$.
h. $v(7) = 9$.
i. $v(8) =$ *8 or 9.*
j. $v(9) =$ *8 or 9.*
k. $v(10) = 10$.

It is interesting to note that $v(g)$ is not monotone.

6. Inequalities for G with v fixed

.

In this section, we study convex lattice polygons, K, with v vertices. Let G denote the number of lattice points inside or on K. We let $G(v) = \inf\{G(K)|v(K) = v\}$ and $g(v) = \inf\{g(K)|v(K) = v\}$.

We can expand a lattice polygon by any amount (by an integer scaling about the origin) keeping v fixed and making G get as large as we want. This gives us the following proposition.

Proposition 6.1. *If v is fixed, then G can be arbitrarily large.*

Proposition 6.2. *If K is a convex lattice polygon with v fixed and smallest G, then K is lean.*

Proof. Suppose P is a lattice point on side A_2A_3 of polygon $A_1A_2A_3A_4A_5 \dots A_v$. Then polygon $A_1PA_3A_4A_5 \dots A_v$ would have the same number of vertices but smaller G. $\square$

Corollary 6.3. $G(v) = g(v) + v$.

So now we know what the smallest G can be. We look at the cases of equality.

Proposition 6.4. *If $v = 3$ then $G \geq 3$. Equality occurs when and only when K is lattice equivalent to the isosceles right triangle shown in figure 6-1.*

```
o .
o o
```

```
Figure 6-1
Unique convex lattice polygon with v=3
and smallest G
```

From $G(v) = g(v) + v$ and $g(3) = 0$, we see that $G(3) = 3$. The uniqueness of figure 6-1 follows from Census-0.

Proposition 6.5. *If $v = 4$ then $G \geq 4$. Equality occurs when and only when K is lattice equivalent to the unit square shown in figure 6-2.*

```
o o
o o
```

```
Figure 6-2
Unique convex lattice polygon with v=4
and smallest G
```

From $G(v) = g(v) + v$ and $g(4) = 0$, we see that $G(4) = 4$. The uniqueness of figure 6-2 follows from Census-0.

Proposition 6.6. *If $v = 5$ then $G \geq 6$. Equality occurs when and only when K is lattice equivalent to the pentagon shown below.*

```
. o .
o . o
o o .
```

```
Figure 6-3
Unique convex lattice polygon with v=5
and smallest G
```

From $G(v) = g(v) + v$ and $g(5) = 1$, we see that $G(5) = 6$. The uniqueness of figure 6-3 follows from Proposition 2.3.

Proposition 6.7. *If $v = 6$ then $G \geq 7$. Equality occurs when and only when K is lattice equivalent to the centrally symmetric hexagon shown below.*

```
. o o
o . o
o o .
```

Figure 6-4
Unique convex lattice polygon with v=6
and smallest G

From $G(v) = g(v) + v$ and $g(6) = 1$, we see that $G(6) = 7$. The uniqueness of figure 6-4 follows from The Central Hexagon Theorem.

Lemma 6.8. *If K is a convex lattice polygon with $v = 7$ and $g = 4$, then no three interior lattice points can colline.*

Proof. Suppose X, Y, and Z are three interior lattice points that all lie on line L. If both sides of L each contain three vertices of K, then let A, B, and C be three vertices on the side of L that does not contain the fourth interior lattice point (figure 6-5a). Thus convex lattice pentagon $ABCZX$ is lattice-point-free, a contradiction.

Figure 6-5a and 6-5b and 6-5c

If one side of L contained four vertices of K, A, B, C, and D, (figure 6-5b), then hexagon $ABCDZX$ would contain at most one interior lattice point, contradicting the Fat Hexagon Theorem.

The only other case is that L passes through two vertices of K, say A and E and one side of L contains three vertices of K, say B, C, and D (figure 6-5c). Then pentagon $ABCDE$ would contain at most one interior lattice point, contradicting the Fat Pentagon Theorem. $\square$

Lemma 6.9. *If $v = 7$ and $g = 4$, then the 4 interior lattice points form a parallelogram.*

Proof. By Lemma 6.8, no three of the interior lattice points colline, so they form a polygon. This polygon cannot be a triangle by the Interior Hull Vertex Inequality. This polygon cannot be a trapezium by the Lattice Trapezium Theorem. It cannot be a (proper) trapezoid by Census-0. Hence it must be a parallelogram. □

Proposition 6.10. *If $v = 7$ then $G \geq 11$. Equality occurs when and only when K is lattice equivalent to one of the heptagons shown below.*

```
. o . .        . o o .
o . . o        . . . o
o . . o        o . . o
. o o .        o o . .
```

Figure 6-6
Only convex lattice polygons with v=7
and smallest G

Proof. From $G(v) = g(v) + v$ and $g(7) = 4$, we see that $G(7) = 11$. If $v = 7$ and $G = 11$, then $g = 4$. By Lemma 6.9, the 4 interior lattice points form a parallelogram. Applying an appropriate integral unimodular affine transformation, we can map these 4 lattice points into a square. They are shown by plus signs in figure 6-7.

```
a b c d
n + + e
m + + f
k j h g
```

Figure 6-7

By the Outer Parallel Condition, the vertices of K must be 7 of the 12 lattice points labelled a-n in Figure 6-7. If no corner lattice point (a, d, g, or k) belongs to K, then we must choose 7 out of 8 remaining vertices. This gives us a figure that is lattice equivalent to the first polygon shown in figure 6-6. If one corner lattice point belongs to K, say point k, then the two adjacent lattice points, j and m, must also be vertices of K and it is then easy to see that the polygon must be lattice equivalent to the second polygon shown in figure 6-6. □

Proposition 6.11. *If $v = 8$ then $G \geq 12$. Equality occurs when and only when K is lattice equivalent to the centrally symmetric octagon shown below.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 6-8
Unique convex lattice polygon with v=8
and smallest G

From $G(v) = g(v) + v$ and $g(8) = 4$, we see that $G(8) = 12$. The uniqueness of figure 6-8 follows from The Central Octagon Theorem.

Proposition 6.12. *If $v = 9$ then $G \geq 16$. Equality occurs when and only when K is lattice equivalent to the nonagon shown below.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

```
Figure 6-9
Unique convex lattice polygon with v=9
and smallest G
```

From $G(v) = g(v) + v$ and $g(9) = 7$, we see that $G(9) = 16$. The uniqueness of figure 6-9 follows from Proposition 4.17.

From $G(v) = g(v)+v$ and $g(10)$ =9 or 10, we see that $G(10)$ is 19 or 20. The following result would then follow from Conjecture 4.20.

Conjecture 6.13. *If $v = 10$ then $G \geq 20$. Equality occurs when and only when K is lattice equivalent to the decagon shown below.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

```
Figure 6-10
Conjectured unique convex lattice polygon with v=10
and smallest G
```

Proposition 6.14. *The function $G(v)$ is monotone.*

This follows from the fact that $G(v) = g(v) + v$ and both $g(v)$ and v are monotone (non-decreasing).

We may summarize this data as follows:

Theorem 6.15. *Let $G(v) = \inf\{G(K)|v(K) = v\}$. Then*

a. $G(3) = 3$.
b. $G(4) = 4$.
c. $G(5) = 6$.
d. $G(6) = 7$.
e. $G(7) = 11$.
f. $G(8) = 12$.
g. $G(9) = 16$.
h. $G(10) =$ *19 or 20.*
i. $G(v) = g(v) + v$.

References

[1] J. R. Arkinstall, "Minimal Requirements for Minkowski's Theorem in the Plane I", *Bulletin of the Australian Mathematical Society*. **22**(1980)259–274.

[2] Donald B. Coleman, "Stretch: A Geoboard Game", *Mathematics Magazine*. **51**(1978)49–54.

[3] Stanley Rabinowitz, *Convex Lattice Polytopes, Ph. D. Dissertation*. Polytechnic University. Brooklyn, NY: 1986.

[4] Stanley Rabinowitz, "A Census of Convex Lattice Polygons with at Most One Interior Lattice Point", *Journal of Combinatorial Mathematics and Combinatorial Computing*. **3**(1989)??-??.

[5] P. R. Scott, "On Convex Lattice Polygons", *Bulletin of the Australian Mathematical Society*. **15**(1976)395-399.

[6] Nicole Steinberg, *The Octagon Anomaly*. Westinghouse Paper. Brooklyn, NY: 1988.