Oblique Pythagorean Lattice Triangles

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A *lattice point* is a point in the plane with integer coordinates. A *lattice triangle* is a triangle whose vertices are lattice points. A *Pythagorean triangle* is a right triangle with integer sides.

It is obvious that, given any Pythagorean triangle, a congruent copy can be found in the lattice with its legs parallel to the coordinate axes.

Definition. A triangle is *oblique* (or is embedded in an oblique manner), if no side is parallel to one of the coordinate axes.

In general, given a Pythagorean triangle (such as a 3-4-5 triangle), it is not possible to find a congruent copy embedded obliquely in the lattice. The author asked in this journal ([3]) if there is an oblique lattice triangle similar to a 3-4-5 right triangle. A solution was given in [1]. In this note, we will investigate this question in more detail.

A computer search finds that the smallest oblique lattice triangle similar to a 3-4-5 triangle has vertices at $(0, 0)$, $(4, 4)$, and $(7, 1)$. This triangle is shown in figure 1.

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. . . . o . . .
. . . . . . . .
. . . . . . . .
. . . . . . . o
o . . . . . . .
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figure 1

Note that the sides of this triangle have lengths $3\sqrt{2}$, $4\sqrt{2}$, and $5\sqrt{2}$. A more interesting question is: Can such a triangle have integral sides? The answer is "yes" as we will see below.

We can find an entire family of lattice triangles similar to the 3-4-5 triangle by considering the three points:

$$O = (0, 0)$$
$$B = (4m, 4n)$$
$$C = (4m + 3n, 4n - 3m)$$

where m and n are any positive integers. Note that letting $m = 1$ and $n = 1$ yields the triangle previously found by the computer search.

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To make the sides of the triangle integral, first make OB integral. To do this, apply the general formula for the sides of a Pythagorean triangle: let $m = p^2 - q^2$ and $n = 2pq$. This yields the 2-parameter solution

$$\begin{aligned} O &= (0,0) \\ B &= (4p^2 - 4q^2, 8pq) \\ C &= (4p^2 - 4q^2 + 6pq, 8pq - 3p^2 + 3q^2). \end{aligned}$$

In some of these, a side may be parallel to one of the axes. It is simple to avoid such a case. For example, choose $p = 2$ and $q = 1$ to get the integral triangle with vertices at $(0,0)$, $(12,16)$, and $(24,7)$. This triangle has sides of lenghts 15, 20, and 25. Its sides are 5 times as large as the sides of a 3-4-5 triangle. A computer search reveals that this is the smallest integral triangle similar to a 3-4-5 triangle with no side parallel to an axis.

We now show that this can be done in general.

Theorem 1. *Given a Pythagorean Triangle, one can find an oblique Pythagorean lattice triangle similar to the given triangle.*

Proof. Suppose the given Pythagorean triangle has sides r, s, and t, with t being the length of the hypotenuse. Let $A = (m,n)$. Lay off r copies of OA along ray OA to bring us to the point $B = (rm, rn)$. Erect a perpendicular to OB at B and lay off s copies of OA to bring us to the point $C = (rm - sn, rn + sn)$.

Now let $m = p^2 - q^2$ and $n = 2pq$ to guarantee that OA has integral length. Then we have constructed a Pythagorean triangle OBC similar to the given triangle. Sides OB and BC are clearly not parallel to any axis. OC might be parallel to the y-axis. To prevent this, take $p = 4s$ and $q = 1$. Then the sides of the resulting triangle are:

$$\begin{aligned} O &= (0,0) \\ B &= (16rs^2 - r, 8sr) \\ C &= (16rs^2 - r - 8s^2, 8rs + 8s^2). \end{aligned}$$

The line OC cannot be parallel to the y-axis, since that would require $16rs^2 = r + 8s^2$ or $s^2 = r/8(2r-1) \leq (2r-1)/8(2r-1) = 1/8$, which cannot be since s^2 is a positive integer.

Recall that a Pythagorean Triangle is called *primitive* if its three sides are relatively prime.

The above procedure always produces a non-primitive Pythagorean triangle, since all sides of the triangle formed are divisible by the length of OA and it is clear that $OA > 1$. It is therefore natural to ask if there is a primitive Pythagorean triangle embedded obliquely in the lattice. We answer this question in the negative.

Theorem 2. *No primitive Pythagorean triangle can be embedded obliquely in the lattice.*

Proof. Suppose Pythagorean triangle ABC (with right angle at C) is embedded obliquely in the lattice. Translate the triangle so that C coincides with the origin. Then perform a rotation through a multiple of $\pi/2$ until ray CB lies in the first quadrant. Point B will not be mapped onto an axis since the triangle is still embedded obliquely (and this property is not affected by the translations or rotations just performed). We may assume that point A has been moved into the second quadrant, for if it moved into the third quadrant, we may perform a reflection about the line $y = x$ to bring it into the second quadrant, leaving B in the first quadrant. Furthermore, we may assume that B lies further from the x-axis than A, for if A were further from the x-axis, we could perform a reflection about the y-axis and then relabel points A and B. Thus, $\triangle ABC$ is situated as shown in figure 2.

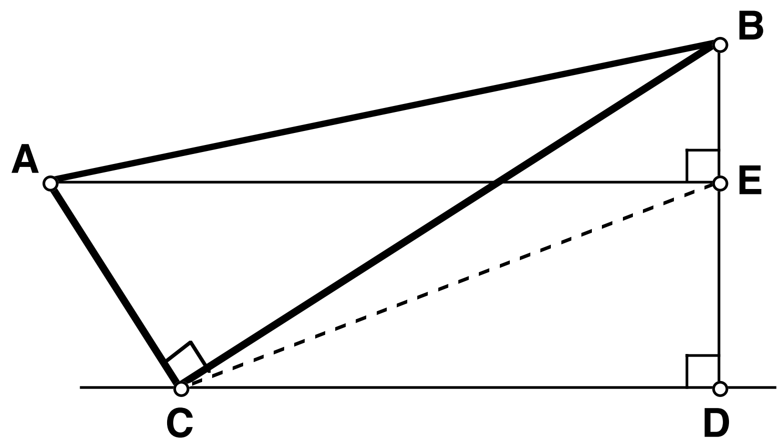

Figure 2

Let D be the foot of the perpendicular from B to the x-axis, and let E be the foot of the perpendicular from A to BD. Since B was further from the x-axis than A, point E lies between B and D. Also note that since A and B are lattice points, the coordinates of points A, B, D, and E are integers. Quadrilateral $ACEB$ is cyclic since $\angle ACB = \angle AEB = \pi/2$. Thus, $\angle ABC = \angle AEC$. But $AE \parallel CD$ implies that $\angle AEC = \angle ECD$. Thus $\angle ABC = \angle ECD$. But triangles ECD and ABC are right triangles. Hence they are similar. Let the ratio of similarity be p/q with $\gcd(p, q) = 1$. This ratio is rational since it is equal to the ratio of DE to AC, both of which are integral. But $AB > BC > CE$, so $\triangle ABC$ is strictly larger than $\triangle CDE$, and so $q > 1$. Now $CE = (p/q) \cdot AB$, so CE is rational. But $CE^2 = CD^2 + DE^2$, so CE^2 is an integer. If a rational number squared is integral, the rational number must itself be an integer. Hence CE is an integer. Let the lengths of the sides of $\triangle ABC$ be a, b, and c. Then the lengths of the sides of $\triangle ECD$ are pa/q, pb/q, and pc/q. But these lengths are integers and p and q are relatively prime. So $q|a$, $q|b$, and $q|c$. Thus, $q| \gcd(a, b, c)$ and consequently, $\triangle ABC$ is not primitive.

Corollary. *The set of diophantine equations*

$$\begin{aligned} a^2 + b^2 &= r^2 \\ (b+d)^2 + c^2 &= s^2 \\ (a+c)^2 + d^2 &= t^2 \\ r^2 + s^2 &= t^2 \end{aligned}$$

has no solution with r, s, and t being relatively prime.

Proof. In the preceding configuration, let point B have coordinates (c, d), let C have coordinates $(-a, b+d)$ and let $AC = r$, $AB = s$, and $BC = t$. Now the above equations represent the Pythagorean Theorem applied to the various right triangles involved.

Although no oblique lattice triangle congruent to the 3-4-5 triangle exists in the plane lattice, what about in the higher dimensions? We conclude this paper with the following surprise: An oblique 3-4-5 triangle exists in the integer lattice in 7-dimensional space! Its vertices are given by the points

$$\begin{aligned} O &= (0,0,0,0,0,0,0) \\ B &= (1,2,2,0,0,0,0) \\ C &= (0,0,0,2,2,2,2). \end{aligned}$$

For other easily-stated but unsolved problems concerning lattice points, consult [2].

REFERENCES

[1] Charles R. Diminnie, Richard I. Hess, and John Putz, "Solution to Problem 581", *Pi Mu Epsilon Journal.* **8**(1985)194.

[2] J. Hammer, *Unsolved Problems Concerning Lattice Points, Research Notes in Mathematics, No. 15.* Pitman. London: 1978.

[3] Stanley Rabinowitz, "Problem 581", *Pi Mu Epsilon Journal.* **8**(1984)43.