The Factorization of $x^5 \pm x + n$

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It is a surprising fact that $x^5 - x + 2759640$ factors as the product

$$(x^2 + 12x + 377) \times (x^3 - 12x^2 - 233x + 7320).$$

In fact, the quintic,

$$x^5 \pm x + n, \tag{1}$$

rarely factors at all. It is the purpose of this note to find all n for which (1) is reducible.

Clearly, (1) has the linear factor $x + a$ if and only if n is of the form $a^5 \pm a$. So we are more interested in the question of when does (1) factor as the product of a quadratic polynomial and a cubic polynomial.

Assuming that we have the factorization

$$x^5 + mx + n = (x^2 + ax + b)(x^3 + cx^2 + dx + e),$$

we can equate like powers of x in succession to find:

$$c = -a$$
$$d = a^2 - b$$
$$e = a(2b - a^2)$$
$$m = 3a^2b - a^4 - b^2 \tag{2}$$

and

$$n = ab(2b - a^2). \tag{3}$$

Eliminating b from (2) and (3) yields

$$n^2 + (4am - 11a^5)n + a^2(m + a^4)(4m - a^4) = 0.$$

This is a quadratic in n whose solution is

$$n = \frac{11a^5 - 4am \pm 5a^3\sqrt{5a^4 - 4m}}{2}. \tag{4}$$

In order for n to be integral, we must have $5a^4 - 4m = z^2$ for some integral z. Since we are interested in the cases where $m = \pm 1$, we are looking to solve the Diophantine equation $z^2 - 5a^4 = \pm 4$. Let $x = a^2$. Note that x and z must have the same parity, so that we may let $y = (x + z)/2$, where y is also an integer. This puts the equation in the form

$$(2y - x)^2 - 5x^2 = \pm 4$$

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or equivalently,

$$y^2 - xy - x^2 = \pm 1 \tag{5}$$

where it is desired that x be a perfect square.

Equation (5) brings to mind a known property of Fibonacci numbers, namely that the integer x is a Fibonacci number if and only if there is an integer y such that $y^2 - xy - x^2 = \pm 1$. (This is proven in [3] and [4].) Thus we see that $x = a^2$ must be a Fibonacci number.

But it is also known that the only square Fibonacci numbers are 0, 1, and 144 (see [1] or [2]). If $a^2 = 0$, then $n = 0$ and several trivial factorizations are possible. These will be excluded from the following discussion. Let us now consider the two cases, $m = +1$ and $m = -1$.

Case 1: $m = +1$.

If $a^2 = 1$, then $a = \pm 1$ and and using (4) to find n gives $n = \pm 1$ or $n = \pm 6$. If $a^2 = 144$ then $a = \pm 12$, but the values of n obtained do not make $5a^4 - 4$ a perfect square so are ruled out.

Case 2: $m = -1$.

If $a^2 = 1$, then $a = \pm 1$ and $n = 0$ or $n = \pm 15$. If $a^2 = 144$, then $a = \pm 12$ and $n = \pm 22440$ or $n = \pm 2759640$.

We can summarize our results by the following theorems:

Theorem 1: The only integral n for which $x^5 + x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 1$ and $n = \pm 6$. The factorizations are

$$x^5 + x \pm 1 = (x^2 \pm x + 1)(x^3 \mp x^2 \pm 1)$$
$$x^5 + x \pm 6 = (x^2 \pm x + 2)(x^3 \mp x^2 - x \pm 3).$$

Theorem 2: The only integral n for which $x^5 - x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 15$, $n = \pm 22440$, and $n = \pm 2759640$. The factorizations are

$$x^5 - x \pm 15 = (x^2 \pm x + 3)(x^3 \mp x^2 - 2x \pm 5)$$
$$x^5 - x \pm 22440 = (x^2 \mp 12x + 55)(x^3 \pm 12x^2 + 89x \pm 408)$$
$$x^5 - x \pm 2759640 = (x^2 \pm 12x + 377)(x^3 \mp 12x^2 - 233x \pm 7320).$$

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