Algorithmic Summation of Reciprocals of Products of Fibonacci Numbers

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1. Introduction.

There is no known simple form for the following summations:

$$\mathbb{F}_N = \sum_{n=1}^{N} \frac{1}{F_n}, \qquad \mathbb{G}_N = \sum_{n=1}^{N} \frac{(-1)^n}{F_n}, \qquad \text{and} \qquad \mathbb{K}_N = \sum_{n=1}^{N} \frac{1}{F_n F_{n+1}}. \tag{1}$$

It is our purpose to show that all other indefinite summations of reciprocals of products of Fibonacci numbers can be expressed in terms of these forms. More specifically, we will give an algorithm for expressing

$$S_N(a_1, a_2, \ldots, a_r) = \sum_{n=1}^{N} \frac{1}{F_{n+a_1} F_{n+a_2} \cdots F_{n+a_r}} \tag{2}$$

and

$$T_N(a_1, a_2, \ldots, a_r) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a_1} F_{n+a_2} \cdots F_{n+a_r}} \tag{3}$$

in terms of $\mathbb{F}_N$, $\mathbb{G}_N$, and $\mathbb{K}_N$, where a_1, $a_2, \ldots, a_r$ are distinct integers. Since a_1, $a_2, \ldots, a_r$ are constants, these symbols may appear in the limits of the summations, but it is our objective to find formulas in which N does not appear in any of the summation limits.

Expressions of the form $S_N(a_1, a_2, \ldots, a_r)$ and $T_N(a_1, a_2, \ldots, a_r)$ will be called reciprocal sums of order r. Those of the second form are also called alternating reciprocal sums.

Without loss of generality, we may assume that the a_i are ordered so that $a_1 < a_2 < \cdots < a_r$. Furthermore, we may assume that $a_1 = 0$, because a change of the index of summation allows us to compute those sums where $a_1 \neq 0$. For example, if $a_1 > 0$, then we have

$$S_N(a_1, a_2, \ldots, a_r) = S_{N+a_1}(0, a_2 - a_1, \ldots, a_r - a_1) - S_{a_1}(0, a_2 - a_1, \ldots, a_r - a_1).$$

2. Reduction Formulas.

We start by showing that reciprocal sums of order r can be expressed in terms of reciprocal sums of order $r - 2$ for all integers $r > 2$.

The following identity is straightforward to prove (for example, by using algorithm `FibSimplify` from [8]):

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Theorem 1 (The Partial Fraction Decomposition Formula).
Let a, b, and c be distinct integers. Then for all integers n,

$$\frac{(-1)^n}{F_{n+a}F_{n+b}F_{n+c}} = \frac{A}{F_{n+a}} + \frac{B}{F_{n+b}} + \frac{C}{F_{n+c}} \tag{4}$$

where

$$A = \frac{(-1)^a}{F_{b-a}F_{c-a}}, \qquad B = \frac{(-1)^b}{F_{c-b}F_{a-b}}, \qquad \text{and} \qquad C = \frac{(-1)^c}{F_{a-c}F_{b-c}}. \tag{5}$$

Theorem 2 (The Reduction Algorithm). If $r > 2$, then any reciprocal sum of order r can be expressed in terms of reciprocal sums of order $r-2$.

Proof: If $f(n)$ is any expression involving n, we see from Theorem 1 that

$$\sum_{n=1}^{N} \frac{1}{f(n)F_{n+a}F_{n+b}F_{n+c}} = \sum_{n=1}^{N} \frac{A(-1)^n}{f(n)F_{n+a}} + \sum_{n=1}^{N} \frac{B(-1)^n}{f(n)F_{n+b}} + \sum_{n=1}^{N} \frac{C(-1)^n}{f(n)F_{n+c}}, \tag{6}$$

with A, B, and C as given in equation (5). If $f(n)$ is the product of $r-3$ factors, each of the form F_{n+c}, then this shows that a reciprocal sum of order r can be expressed in terms of reciprocal sums of order $r-2$, for any integer $r > 2$. (If $r = 3$, then $f(n) = 1$.) Note that $f(n)$ may contain $(-1)^n$ as a factor to allow us to handle alternating reciprocal sums. $\square$

Since we can repeatedly reduce the order of any reciprocal sum by 2, this shows that any reciprocal sum can be expressed in terms of reciprocal sums of orders 1 and 2.

3. Reciprocal Sums of Order 1.

Any reciprocal sum of order 1 differs by a constant from expressions of the form $\mathbb{F}_{N+c}$ or $\mathbb{G}_{N+c}$. Specifically, if $a > 0$, then

$$\sum_{n=1}^{N} \frac{1}{F_{n+a}} = \sum_{n=1}^{a+N} \frac{1}{F_n} - \sum_{n=1}^{a} \frac{1}{F_n} = \mathbb{F}_{N+a} - \mathbb{F}_a \tag{7}$$

and

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = \sum_{n=1}^{a+N} \frac{(-1)^n}{F_n} - \sum_{n=1}^{a} \frac{(-1)^n}{F_n} = \mathbb{G}_{N+a} - \mathbb{G}_a. \tag{8}$$

Thus, reciprocal sums of order 1 are readily computed in terms of $\mathbb{F}$'s and $\mathbb{G}$'s.

4. Alternating Reciprocal Sums of Order 2.

As has been pointed out, for reciprocal sums of order 2, we may assume that the denominator is of the form F_nF_{n+a} with $a > 0$ for if not, the reciprocal sum differs by only a finite number of terms from one of this form.

There are two cases to consider, depending on whether the reciprocal sum is alternating or not.

In the alternating case, an explicit closed form can be found. The following result was proven by Brousseau [3] and Carlitz [5].

Theorem 3 (Computation of Alternating Reciprocal Sums of Order 2).
If $a > 0$, then

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_nF_{n+a}} = \frac{1}{F_a}\left[\sum_{j=1}^{a} \frac{F_{j-1}}{F_j} - \sum_{j=1}^{a} \frac{F_{j+N-1}}{F_{j+N}}\right]. \tag{9}$$

Good [6] has found a different, but equivalent, expression for this reciprocal sum. He has shown that for $a > 0$,

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_nF_{n+a}} = \frac{F_N}{F_a}\sum_{n=1}^{a} \frac{(-1)^n}{F_nF_{n+N}}. \tag{10}$$

Another equivalent formulation is the following. We omit the proof.

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_nF_{n+a}} = \frac{1}{F_a}\left[\sum_{j=1}^{a} \frac{F_{j+1}}{F_j} - \sum_{j=1}^{a} \frac{F_{j+N+1}}{F_{j+N}}\right]. \tag{11}$$

5. Non-alternating Reciprocal Sums of Order 2.

We start with a preliminary result.

Theorem 4. Let H_n be any sequence of nonzero terms that satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$. If $b \geq 0$, then

$$\sum_{n=1}^{N} \frac{1}{H_{n+b}H_{n+b+2}} = \frac{1}{H_{b+1}H_{b+2}} - \frac{1}{H_{N+b+1}H_{N+b+2}}. \tag{12}$$

Proof: We have

$$\frac{1}{H_{n+b}H_{n+b+2}} = \frac{H_{n+b+1}}{H_{n+b}H_{n+b+1}H_{n+b+2}} = \frac{H_{n+b+2}-H_{n+b}}{H_{n+b}H_{n+b+1}H_{n+b+2}}$$
$$= \frac{1}{H_{n+b}H_{n+b+1}} - \frac{1}{H_{n+b+1}H_{n+b+2}}.$$

Summing from 1 to N, we find that the right-hand side telescopes, and we get the desired result. $\square$

Theorem 5. For $a > 0$, let

$$\mathbb{F}_N(a) = \sum_{n=1}^{N} \frac{1}{F_n F_{n+a}}. \tag{13}$$

If we can find a closed form expression for $\mathbb{F}_N(a-2)$, then we can also find a closed form expression for $\mathbb{F}_N(a)$.

Proof: The following identity is well known (see equation (9) in [3]):

$$F_a F_{n+a-2} - F_{a-2}F_{n+a} = (-1)^a F_n. \tag{14}$$

Thus, we find that

$$\frac{F_a}{F_n F_{n+a}} - \frac{F_{a-2}}{F_n F_{n+a-2}} = \frac{(-1)^a}{F_{n+a-2}F_{n+a}}.$$

If we now sum as n goes from 1 to N, we get

$$F_a \mathbb{F}_N(a) - F_{a-2}\mathbb{F}_N(a-2) = (-1)^a \sum_{n=1}^{N} \frac{1}{F_{n+a-2}F_{n+a}}.$$

Applying Theorem 4 gives

$$F_a \mathbb{F}_N(a) - F_{a-2}\mathbb{F}_N(a-2) = (-1)^a \left[\frac{1}{F_{a-1}F_a} - \frac{1}{F_{N+a-1}F_{N+a}}\right]. \tag{15}$$

Solving for $\mathbb{F}_N(a)$ gives

$$\mathbb{F}_N(a) = \frac{F_{a-2}}{F_a}\mathbb{F}_N(a-2) + \frac{(-1)^a}{F_a}\left[\frac{1}{F_{a-1}F_a} - \frac{1}{F_{N+a-1}F_{N+a}}\right] \tag{16}$$

which shows that we can find $\mathbb{F}_N(a)$ if we know $\mathbb{F}_N(a-2)$. $\square$

By induction, we see that any expression of the form

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+a}}$$

with $a > 0$, can be expressed in terms of either

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+1}} \qquad \text{or} \qquad \sum_{n=1}^{N} \frac{1}{F_n F_{n+2}}.$$

The first form is known as $\mathbb{K}_N$. The second form is easily evaluated by setting $b = 0$ in Theorem 4 to get

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+2}} = 1 - \frac{1}{F_{N+1} F_{N+2}}. \tag{17}$$

We have just shown how to find a formula for any reciprocal sum of order 2 in terms of $\mathbb{K}_n$.

We can also find a more explicit formula. If we let $a = 2c+1$ in formula (15), we get

$$F_{2c+1}\mathbb{F}_N(2c+1) - F_{2c-1}\mathbb{F}_N(2c-1) = (-1)^{2c+1}\left[\frac{1}{F_{2c}F_{2c+1}} - \frac{1}{F_{N+2c}F_{N+2c+1}}\right]. \tag{18}$$

Now sum as c goes from 1 to a. The left side telescopes, and we get

$$F_{2a+1}\mathbb{F}_N(2a+1) - \mathbb{K}_N = \sum_{c=1}^{a}\left[\frac{1}{F_{N+2c}F_{N+2c+1}} - \frac{1}{F_{2c}F_{2c+1}}\right]$$

so that

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+2a+1}} = \frac{1}{F_{2a+1}}\left\{\mathbb{K}_N + \sum_{c=1}^{a}\left[\frac{1}{F_{N+2c}F_{N+2c+1}} - \frac{1}{F_{2c}F_{2c+1}}\right]\right\}. \tag{19}$$

Similarly, if $a = 2c$, we can sum as c goes from 1 to a to get

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+2a}} = \frac{1}{F_{2a}}\sum_{c=1}^{a}\left[\frac{1}{F_{2c-1}F_{2c}} - \frac{1}{F_{N+2c-1}F_{N+2c}}\right]. \tag{20}$$

We can summarize these results with the following theorem.

Theorem 6. If a is a positive integer, then

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+a}} = \begin{cases} \dfrac{1}{F_a}\displaystyle\sum_{i=1}^{\lfloor a/2 \rfloor}\left(\frac{1}{F_{N+2i}F_{N+2i+1}} - \frac{1}{F_{2i}F_{2i+1}}\right) + \frac{\mathbb{K}_N}{F_a}, & \text{if } a \text{ is odd,} \\ \dfrac{1}{F_a}\displaystyle\sum_{i=1}^{a/2}\left(\frac{1}{F_{2i-1}F_{2i}} - \frac{1}{F_{N+2i-1}F_{N+2i}}\right), & \text{if } a \text{ is even.} \end{cases} \tag{21}$$

These formulas give us the following values for $\mathbb{F}_N(a)$ for small a:

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+3}} = \frac{1}{2}\left[\mathbb{K}_N + \frac{1}{F_{N+2}F_{N+3}} - \frac{1}{2}\right] \tag{22}$$

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+4}} = \frac{1}{3}\left[\frac{7}{6} - \frac{1}{F_{N+1}F_{N+2}} - \frac{1}{F_{N+3}F_{N+4}}\right] \tag{23}$$

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+5}} = \frac{1}{5}\left[\mathbb{K}_N + \frac{1}{F_{N+2}F_{N+3}} + \frac{1}{F_{N+4}F_{N+5}} - \frac{17}{30}\right] \tag{24}$$

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+6}} = \frac{1}{8}\left[\frac{143}{120} - \frac{1}{F_{N+1}F_{N+2}} - \frac{1}{F_{N+3}F_{N+4}} - \frac{1}{F_{N+5}F_{N+6}}\right]. \tag{25}$$

As $N \to \infty$ in formula (21), we get

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a}} = \begin{cases} \dfrac{1}{F_a}\mathbb{K} - \dfrac{1}{F_a}\displaystyle\sum_{i=1}^{\lfloor a/2 \rfloor} \frac{1}{F_{2i}F_{2i+1}}, & \text{if } a \text{ is odd,} \\ \dfrac{1}{F_a}\displaystyle\sum_{i=1}^{a/2} \frac{1}{F_{2i-1}F_{2i}}, & \text{if } a \text{ is even.} \end{cases} \tag{26}$$

where $\mathbb{K} = \lim_{n\to\infty} \mathbb{K}_n$. For small values of a, these formulas yield the results found by Brousseau in [3].

6. Summary.

We have just shown that any reciprocal sum of order 1 can be expressed in terms of $\mathbb{F}_N$ and $\mathbb{G}_N$; and that any reciprocal sum of order 2 can be expressed in terms of $\mathbb{K}_N$. Thus, we can conclude that all reciprocal sums are expressible in terms of $\mathbb{F}_N$, $\mathbb{G}_N$, and $\mathbb{K}_N$. We also have presented a mechanical algorithm for finding all such representations.

Open Question 1. Is there a simple algebraic relationship between $\mathbb{L}_n = \displaystyle\sum_{n=1}^{N} \frac{1}{L_n}$ and any of $\mathbb{F}_n$, $\mathbb{G}_n$, and $\mathbb{K}_n$?

Open Question 2. Can we find the value of $\displaystyle\sum_{n=1}^{N} \frac{1}{F_n^2}$?

7. Going to Infinity.

If we take the limit as N goes to infinity, we can express many infinite sums in terms of

$$\mathbb{F} = \sum_{n=1}^{\infty} \frac{1}{F_n}, \qquad \mathbb{G} = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}, \qquad \mathbb{K} = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}},$$
$$\mathbb{L} = \sum_{n=1}^{\infty} \frac{1}{L_n}, \qquad \text{and} \qquad \mathbb{J} = \sum_{n=1}^{\infty} \frac{(-1)^n}{L_n}. \tag{27}$$

No simple expressions for these infinite sums are known, however, they have been expressed in terms of Elliptic Functions [4], Theta Series [7], [1], and Lambert Series [2].

For example, we get results of Brousseau [3] such as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+a}} = \frac{1}{F_a} \left[\sum_{j=1}^{a} \frac{F_{j-1}}{F_j} - \frac{a}{\alpha} \right] \tag{28}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5} F_{n+6} F_{n+7} F_{n+8}} = \frac{319}{16380} \left(\mathbb{F} - \frac{46816051}{13933920} \right). \tag{29}$$

Carlitz has also found some pretty results for certain r-th order reciprocal sums in terms of Fibonomial coefficients (see formulas (5.6), (5.7), and (6.7) in [5]).

Open Question 3. Are any of $\mathbb{F}$, $\mathbb{G}$, $\mathbb{K}$, $\mathbb{L}$, $\mathbb{J}$ connected by a simple algebraic relation?

References

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Nag [9] has pointed out that equation (8) is incorrect. The correct result is the following.

If $a > 1$, then

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = (-1)^a \left[\mathbb{G}_{N+a} - \mathbb{G}_a\right]. \tag{8$'$}$$

Proof: Letting $k = n + a$ gives

$$\begin{aligned}
\sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} &= \sum_{k=a+1}^{a+N} \frac{(-1)^{k-a}}{F_k} \\
&= (-1)^a \sum_{k=a+1}^{a+N} \frac{(-1)^k}{F_k} \\
&= (-1)^a \left[\sum_{k=1}^{a+N} \frac{(-1)^k}{F_k} - \sum_{k=1}^{a} \frac{(-1)^k}{F_k}\right] \\
&= (-1)^a \left[\mathbb{G}_{N+a} - \mathbb{G}_a\right].
\end{aligned}$$

Reference

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