Algorithmic Simplification of Reciprocal Sums

Stanley Rabinowitz
12 Vine Brook Road
Westford, MA 01886 USA

1. Introduction.

Algorithms for evaluating or simplifying sums of the form

$$\sum_{n=1}^{N} \frac{1}{F_{n+a_1} F_{n+a_2} \cdots F_{n+a_r}}$$

where the F_i are Fibonacci numbers and the a_i are integers have been discussed in [13]. It is the goal of this paper to generalize these results to arbitrary second-order linear recurrences.

Consider the second order linear recurrences defined by

$$u_{n+2} = Pu_{n+1} - Qu_n, \qquad u_0 = 0, \quad u_1 = 1, \tag{1}$$

$$v_{n+2} = Pv_{n+1} - Qv_n, \qquad v_0 = 2, \quad v_1 = P, \tag{2}$$

and

$$w_{n+2} = Pw_{n+1} - Qw_n, \qquad w_0,\ w_1 \text{ arbitrary.} \tag{3}$$

Let r_1 and r_2 denote the roots of the characteristic equation $x^2 - Px + Q = 0$. Let

$$D = P^2 - 4Q \qquad \text{and} \qquad e = w_0 w_2 - w_1^2. \tag{4}$$

Throughout this paper, we shall assume that $D \neq 0$, $e \neq 0$, $Q \neq 0$, and $w_n \neq 0$ for $n > 0$.

In the case of the Fibonacci sequence, we showed [13] that all reciprocal sums can be expressed in closed form in terms of

$$\mathbb{F}_N = \sum_{n=1}^{N} \frac{1}{F_n}, \qquad \mathbb{G}_N = \sum_{n=1}^{N} \frac{(-1)^n}{F_n}, \qquad \text{and} \qquad \mathbb{K}_N = \sum_{n=1}^{N} \frac{1}{F_n F_{n+1}}. \tag{5}$$

It is our intent to generalize these results to apply to the sequence $\langle w_n \rangle$.

For the following definitions, let r be a positive integer and let a_1, a_2,..., a_r be distinct nonnegative integers.

Definition 1 (Unit Reciprocal Sum). A unit reciprocal sum of order r is a sum of the form

$$\sum_{n=1}^{N} \frac{1}{w_{n+a_1} w_{n+a_2} \cdots w_{n+a_r}}.$$

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Definition 2 (Q-Reciprocal Sum). A Q-reciprocal sum of order r is a sum of the form

$$\sum_{n=1}^{N} \frac{Q^{kn}}{w_{n+a_1} w_{n+a_2} \cdots w_{n+a_r}}$$

where $k = \lfloor r/2 \rfloor$.

Definition 3 (Reciprocal Sum). A reciprocal sum is a unit reciprocal sum or a Q-reciprocal sum.

Definition 4 (Rational Sum). A rational sum of order r is a sum of the form

$$\sum_{n=1}^{N} \frac{f(x_1, x_2, \ldots, x_s)}{w_{n+a_1} w_{n+a_2} \cdots w_{n+a_r}}$$

where $f(x_1, x_2, \ldots, x_s)$ is a polynomial with each of the variables x_i being of the form w_{n+c_i} or Q^n.

In this paper, we will show that all reciprocal sums of orders 1 and 2 can be expressed in closed form in terms of

$$\mathbb{X}_N = \sum_{n=1}^{N} \frac{1}{w_n}, \qquad \mathbb{Y}_N = \sum_{n=1}^{N} \frac{Q^n}{w_n}, \qquad \text{and} \qquad \mathbb{W}_N = \sum_{n=1}^{N} \frac{1}{w_n w_{n+1}}. \tag{6}$$

We will also show that all Q-reciprocal sums (of any order) can be expressed in closed form in terms of $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$.

Finally, we shall show that if $Q = \pm 1$, then all rational sums can be expressed in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$, where

$$\mathbb{U}_N = \sum_{n=1}^{N} \frac{w_{n+1}}{w_n} \qquad \text{and} \qquad \mathbb{V}_N = \sum_{n=1}^{N} \frac{Q^n w_{n+1}}{w_n}.$$

We shall also present mechanical algorithms for finding these closed forms.

We need the following results.

Theorem 1 (The Representation Theorem). If a, b, and c are integers and $u_{a-b} \neq 0$, then

$$w_{n+c} = \frac{u_{c-b}}{u_{a-b}} w_{n+a} + \frac{u_{c-a}}{u_{b-a}} w_{n+b}. \tag{7}$$

(This expresses w_{n+c} in terms of w_{n+a} and w_{n+b}.)

Proof: The identity can be mechanically verified by using algorithm `LucasSimplify` from [11]. $\square$

Theorem 1 can be put into a more symmetrical form:

Theorem 2. For all integers a, b, and c,

$$Q^c u_{b-c} w_{n+a} + Q^a u_{c-a} w_{n+b} + Q^b u_{a-b} w_{n+c} = 0. \tag{8}$$

(This gives a symmetric connection between w_{n+a}, w_{n+b}, and w_{n+c}.)

Proof: This follows from the Representation Theorem by making use of the well-known Negation Formula [11]: $u_{-n} = -u_n Q^{-n}$. □

Theorem 3. If a, b, and c are integers and $u_{a-b} \neq 0$, then

$$\frac{1}{w_{n+a} w_{n+b}} = \frac{A}{w_{n+c} w_{n+a}} + \frac{B}{w_{n+c} w_{n+b}} \tag{9}$$

where

$$A = \frac{u_{c-a}}{u_{b-a}} \qquad \text{and} \qquad B = \frac{u_{c-b}}{u_{a-b}}.$$

(This allows one to convert reciprocal sums of order 2 to those in which w_{n+c} occurs as a factor of the denominator.)

Proof: This is an immediate consequence of the Representation Theorem. □

2. Reciprocal Sums of Order 1.

There are no known elementary forms for the reciprocal sums of order 1, so we shall give them names:

$$\mathbb{X}_N = \sum_{n=1}^{N} \frac{1}{w_n} \qquad \text{and} \qquad \mathbb{Y}_N = \sum_{n=1}^{N} \frac{Q^n}{w_n}. \tag{10}$$

Strictly speaking, we should write these as $\mathbb{X}_N(w_0, w_1, P, Q)$ and $\mathbb{Y}_N(w_0, w_1, P, Q)$; but we will simply write $\mathbb{X}_N$ and $\mathbb{Y}_N$ when w_0, w_1, P, and Q are fixed.

If $a > 0$, we have

$$\sum_{n=1}^{N} \frac{1}{w_{n+a}} = \mathbb{X}_{N+a} - \mathbb{X}_a \tag{11}$$

and

$$\sum_{n=1}^{N} \frac{Q^n}{w_{n+a}} = \mathbb{Y}_{N+a} - \mathbb{Y}_a. \tag{12}$$

Thus all reciprocal sums of order 1 can be expressed in terms of $\mathbb{X}_N$ and $\mathbb{Y}_N$.

3. Q-Reciprocal Sums of Order 2.

Theorem 4. If $a > 0$, then

$$u_a \sum_{n=1}^{N} \frac{Q^n}{w_n w_{n+a}} = \frac{Q}{e}\left[\sum_{n=1}^{a} \frac{w_{n-1}}{w_n} - \sum_{n=1}^{a} \frac{w_{N+n-1}}{w_{N+n}}\right]. \tag{13}$$

Proof: We begin with the identity

$$w_{n+a}w_{n-1} - w_n w_{n+a-1} = Q^{n-1} e u_a \tag{14}$$

which comes from d'Ocagne's Identity (see [12]). Thus, we have

$$\frac{w_{n-1}}{w_n} - \frac{w_{n+a-1}}{w_{n+a}} = \frac{Q^{n-1} e u_a}{w_n w_{n+a}}.$$

Summing from 1 to N yields

$$u_a \sum_{n=1}^{N} \frac{Q^{n-1}}{w_n w_{n+a}} = \frac{1}{e}\left[\sum_{n=1}^{N} \frac{w_{n-1}}{w_n} - \sum_{n=1}^{N} \frac{w_{n+a-1}}{w_{n+a}}\right] = \frac{1}{e}\left[\sum_{n=1}^{a} \frac{w_{n-1}}{w_n} - \sum_{n=1}^{a} \frac{w_{N+n-1}}{w_{N+n}}\right]$$

which is the desired result. □

This can be put into a more symmetrical form. The following theorem is a generalization of a result by Good [4] and was proven by André-Jeannin [1].

Theorem 5 (Symmetry Property for Reciprocal Sums). If $a > 0$, then

$$u_a \sum_{n=1}^{N} \frac{Q^n}{w_n w_{n+a}} = u_N \sum_{n=1}^{a} \frac{Q^n}{w_n w_{n+N}}. \tag{15}$$

Proof: Again we use d'Ocagne's identity. Putting $a = N$ in formula (14) gives

$$w_{n-1}w_{N+n} - w_n w_{N+n-1} = Q^{n-1} e u_N.$$

Combining the two sums on the right-hand side of Theorem 4 and applying this identity yields the desired result. □

Corollary (letting $a = 1$).

$$\sum_{n=1}^{N} \frac{Q^n}{w_n w_{n+1}} = \frac{Q u_N}{w_1 w_{N+1}} = \frac{Q}{e}\left[\frac{w_0}{w_1} - \frac{w_N}{w_{N+1}}\right]. \tag{16}$$

4. Unit Reciprocal Sums of Order 2.

For non-alternating reciprocal Fibonacci sums, we had to introduce (in [13]) the symbol

$$\mathbb{K}_N = \sum_{n=1}^{N} \frac{1}{F_n F_{n+1}} \tag{17}$$

for a sum with no known simple closed form. In a similar manner, we need to do the same thing for unit reciprocal sums for the sequence $\langle w_n \rangle$.

Let

$$\mathbb{W}_N = \sum_{n=1}^{N} \frac{1}{w_n w_{n+1}} \tag{18}$$

with the understanding that $\mathbb{W}_0 = 0$.

Again, we should really write this as $\mathbb{W}_N(w_0, w_1, P, Q)$; but if $\langle w_n \rangle$ is a fixed sequence, we will simply write this as $\mathbb{W}_N$.

Theorem 6. If $c > 0$ and $u_c \neq 0$, then

$$\sum_{n=1}^{N} \frac{1}{w_n w_{n+c}} = \frac{1}{u_c} \sum_{i=0}^{c-1} Q^i (\mathbb{W}_{N+i} - \mathbb{W}_i). \tag{19}$$

Proof: Letting $a = i$, $b = i + 1$, and $c = 0$ in Theorem 3, we get

$$\frac{u_{i+1}}{w_n w_{n+i+1}} - \frac{u_i}{w_n w_{n+i}} = \frac{Q^i}{w_{n+i} w_{n+i+1}}$$

using the Negation Formula $u_{-n} = -u_n Q^{-n}$. Summing as n goes from 1 to N yields

$$u_{i+1} \mathbb{W}_N(i+1) - u_i \mathbb{W}_N(i) = Q^i (\mathbb{W}_{N+i} - \mathbb{W}_i)$$

where

$$\mathbb{W}_N(a) = \sum_{n=1}^{N} \frac{1}{w_n w_{n+a}}.$$

Now sum as i goes from 0 to $c - 1$. The left side telescopes and we get

$$u_c \mathbb{W}_N(c) = \sum_{i=0}^{c-1} Q^i (\mathbb{W}_{N+i} - \mathbb{W}_i)$$

which gives our desired result. $\square$

Thus, all reciprocal sums of order 2 can be expressed in terms of $\mathbb{W}_N$.

5. The Reduction Process.

We now show how to simplify certain reciprocal sums with three or more factors in the denominator.

Theorem 7 (The Partial Fraction Decomposition Formula for w).
For all n,

$$\frac{Q^n}{w_{n+a}w_{n+b}w_{n+c}} = \frac{A}{w_{n+a}} + \frac{B}{w_{n+b}} + \frac{C}{w_{n+c}} \tag{20}$$

where

$$A = \frac{-Q^{-a}}{eu_{b-a}u_{c-a}}, \qquad B = \frac{-Q^{-b}}{eu_{c-b}u_{a-b}}, \qquad \text{and} \qquad C = \frac{-Q^{-c}}{eu_{a-c}u_{b-c}}. \tag{21}$$

Proof: This result can be mechanically proven using algorithm `LucasSimplify` from [11]. $\square$

Theorem 8 (The Reduction Theorem for w). If $r > 2$, then any Q-reciprocal sum of order r can be expressed in terms of Q-reciprocal sums of order $r-2$.

Proof: By Theorem 7, we have

$$\frac{Q^{kn}}{w_{n+a}w_{n+b}w_{n+c}} = \frac{AQ^{(k-1)n}}{w_{n+a}} + \frac{BQ^{(k-1)n}}{w_{n+b}} + \frac{CQ^{(k-1)n}}{w_{n+c}} \tag{22}$$

where A, B, and C are given in display (21). If $r > 2$ and $k = \lfloor r/2 \rfloor$, then we can take the last three factors in the denominator and apply Theorem 7. This breaks the given sum up into sums with $r-2$ factors in the denominator. The numerators have terms that are constant multiples of $Q^{(k-1)n}$ where $k-1 = \lfloor (r-2)/2 \rfloor$, thus making these sums multiples of Q-reciprocal sums of order $r-2$. $\square$

Corollary. Any Q-reciprocal sum of order r can be expressed in terms of reciprocal sums of order 1 or 2. If $Q = \pm 1$, then any reciprocal sum of order r can be expressed in terms of reciprocal sums of order 1 or 2.

Proof: Apply Theorem 8 repeatedly, until the order of the Q-reciprocal sum becomes 1 or 2. If $Q = \pm 1$, then formula (20) can be written in the form

$$\frac{1}{w_{n+a}w_{n+b}w_{n+c}} = \frac{A(-1)^n}{w_{n+a}} + \frac{B(-1)^n}{w_{n+b}} + \frac{C(-1)^n}{w_{n+c}}. \tag{23}$$

Applying this repeatedly reduces the order of the reciprocal sum to 1 or 2. $\square$

By induction, we can state a more general form of the Partial Fraction Decomposition Theorem.

Theorem 9 (The Generalized Partial Fraction Decomposition Formula). If r is a positive integer, then

$$\frac{1}{w_{n_1}w_{n_2}w_{n_3}\cdots w_{n_{2r+1}}} = \sum_{i=1}^{2r+1} \frac{A_i}{w_{n_i}} \tag{24}$$

where $A_i^{-1} = (-eQ^{n_i})^r \prod_{j\neq i} u_{n_j-n_i}$.

6. The Simplification Algorithm.

We can also handle sums similar to reciprocal sums, but in which the numerators are polynomials in the w's. These are called rational sums.

We need to add in two new primitives:

$$\mathbb{U}_N = \sum_{n=1}^{N} \frac{w_{n+1}}{w_n} \qquad \text{and} \qquad \mathbb{V}_N = \sum_{n=1}^{N} \frac{Q^n w_{n+1}}{w_n}. \tag{25}$$

Once again, these would more properly be written as $\mathbb{U}_N(w_0, w_1, P, Q)$ and $\mathbb{V}_N(w_0, w_1, P, Q)$; but $\mathbb{U}_N$ and $\mathbb{V}_N$ will suffice when the sequence is fixed.

We now show how to evaluate a wide class of reciprocal and rational sums in closed form in terms of the quantities $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$.

Definition. A w-polynomial in the variable n is any polynomial $f(x_1, x_2, \ldots, x_r)$ with constant coefficients where each x_i is of the form w_x or Q^x, with each x of the form $n + c_j$, where the c_j are integer constants. For purposes of this definition, the quantities P, Q, w_0, w_1, and e are to be considered constants.

Theorem 10 (The Simplification Theorem for $Q = \pm 1$). Suppose that P, Q, w_0, and w_1 are fixed constants, thereby determining the sequence $\langle w_n \rangle$. Let $f(n)$ be any w-polynomial in the variable n. For r a positive integer, let c_j, $j = 1, 2, \ldots, r$ be distinct integers. Assume that $w_n > 0$ and $u_n > 0$ for all $n > 0$. Furthermore, if $Q = \pm 1$, then we can find

$$\sum_{n=1}^{N} \frac{f(n)}{w_{n+c_1}w_{n+c_2}\cdots w_{n+c_r}}$$

in closed form in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$.

Proof: As proof, we give the following algorithm.

<u>Algorithm `WReciprocalSum`</u> to evaluate certain reciprocal sums in closed form:

INPUT: A rational sum meeting the conditions of Theorem 9.

OUTPUT: A closed form for the sum expressed in terms of the quantities $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$.

STEP 1: [Reduce the order.] If the denominator consists of three or more terms of the form w_x, choose any three of them, say w_{n+a}, w_{n+b}, and w_{n+c}, and make the following substitution:

$$\frac{1}{w_{n+a}w_{n+b}w_{n+c}} = \frac{AQ^{-n}}{w_{n+a}} + \frac{BQ^{-n}}{w_{n+b}} + \frac{CQ^{-n}}{w_{n+c}} \tag{26}$$

where A, B, and C are given by formula (21). Expand out and make the transformation

$$\sum_{n=1}^{N}[f(n)+g(n)] = \sum_{n=1}^{N} f(n) + \sum_{n=1}^{N} g(n) \tag{27}$$

summing each term on the right by this algorithm.

STEP 2: [Normalize subscripts in denominator.] If the denominator is of the form w_{n+a} or of the form $w_{n+a}w_{n+b}$ with $a \neq 0$ and $a < b$, then apply one of the following transformations:

$$\sum_{n=1}^{N} \frac{f(n)}{w_{n+a}w_{n+b}} = \sum_{n=a+1}^{a+N} \frac{f(n-a)}{w_n w_{n+b-a}} \tag{28}$$

$$\sum_{n=1}^{N} \frac{f(n)}{w_{n+a}} = \sum_{n=a+1}^{a+N} \frac{f(n-a)}{w_n}. \tag{29}$$

STEP 3: [Normalize index of summation.] If the index of summation does not start at 1, then add or subtract a finite number of terms to make the index start at 1. Specifically, if n_0 is a constant and $n_0 \neq 1$, then apply the transformation

$$\sum_{n=n_0}^{N} f(n) = \begin{cases} \displaystyle\sum_{n=1}^{N} f(n) - \sum_{n=1}^{n_0-1} f(n), & \text{if } n_0 > 1, \\ \displaystyle\sum_{n=1}^{N} f(n) + \sum_{n=n_0}^{0} f(n), & \text{if } n_0 \leq 0. \end{cases} \tag{30}$$

STEP 4: [Break up sums.] Expand out the numerator. If the numerator consists of a sum of terms, then sum each term individually. That is, apply the transformation

$$\sum_{n=1}^{N} \frac{f(n)+g(n)}{d} = \sum_{n=1}^{N} \frac{f(n)}{d} + \sum_{n=1}^{N} \frac{g(n)}{d}. \tag{31}$$

In each fraction, cancel any factors of the form w_{n+c} common to the numerator and denominator. Then evaluate each sum recursively using this algorithm. Return the sum of the results so obtained.

STEP 5: [Normalize numerator.] If the denominator is of the form $w_n w_{n+a}$ with $a > 0$ and if the numerator contains a subexpression of the form w_{n+c} where $c \neq 0$ and $c \neq a$, then express this subexpression in terms of w_n and w_{n+a} by using the Representation Theorem. Specifically, make the substitution

$$w_{n+c} = \frac{Q^a u_{c-a}}{u_a} w_n + \frac{u_{c-a}}{u_a} w_{n+a}. \tag{32}$$

If the numerator contains a subexpression of the form Q^n, then express this subexpression in terms of w_n and w_{n+a} by using the formula

$$Q^n = \frac{v_a w_n w_{n+a} - Q^a w_n^2 - w_{n+a}^2}{e u_a^2}. \tag{33}$$

Go back to step 4.

STEP 6: [Normalize numerator (continued).] If the denominator is of the form w_n, and if the numerator contains a subexpression of the form w_{n+c} where $c \neq 0$ and $c \neq 1$, then express this subexpression in terms of w_n and w_{n+1} by using the Representation Theorem. Specifically, make the substitution

$$w_{n+c} = u_c w_{n+1} - Q u_{c-1} w_n. \tag{34}$$

Go back to step 4.

STEP 7: [Evaluate polynomial sums.] If the summand is a w-polynomial in the variable n, evaluate the sum by using algorithm `LucasSum` from [11]. Exit.

STEP 8: [Reduce numerator.] If the denominator is of the form w_n and if the numerator contains a subexpression of the form w_{n+1}^r with $r > 1$, then write w_{n+1}^r as $w_{n+1}^{r-2} w_{n+1}^2$ and reduce the exponent by 1 by applying the substitution

$$w_{n+1}^2 = P w_n w_{n+1} - Q w_n^2 - e Q^n. \tag{35}$$

Expand the numerator. If $Q = -1$, replace any terms of the form Q^{rn+d} by $Q^d (Q^r)^n$. Repeat this step as long as possible, then go back to step 4.

STEP 9: [Pull out constants.] Replace any expressions of the form Q^{n+b} where b is a constant by $Q^b Q^n$. If the numerator is of the form c, cQ^n, $cQ^n w_x$, or $c w_x^r$, where c is a constant ($c \neq 0$ and $c \neq 1$), then apply the transformation

$$\sum_{n=1}^{N} c f(n) = c \sum_{n=1}^{N} f(n). \tag{36}$$

STEP 10: [Handle sums of order 2.] If the denominator is of the form $w_n w_{n+a}$, then evaluate the sum by using one of the following formulas:

$$\sum_{n=1}^{N} \frac{Q^n}{w_n w_{n+a}} = \frac{u_N}{u_a} \sum_{n=1}^{a} \frac{Q^n}{w_n w_{n+N}}; \tag{37}$$

$$\sum_{n=1}^{N} \frac{1}{w_n w_{n+a}} = \frac{1}{u_a} \sum_{i=0}^{a-1} Q^i (\mathbb{W}_{N+i} - \mathbb{W}_i). \tag{38}$$

STEP 11: [Handle basic sums.] If the summand is of one of the following forms, make the substitution shown.

$$\begin{aligned} \sum_{n=1}^{N} \frac{w_{n+1}}{w_n} &= \mathbb{U}_N \\ \sum_{n=1}^{N} \frac{Q^n w_{n+1}}{w_n} &= \mathbb{V}_N \\ \sum_{n=1}^{N} \frac{1}{w_n w_{n+1}} &= \mathbb{W}_N \\ \sum_{n=1}^{N} \frac{1}{w_n} &= \mathbb{X}_N \\ \sum_{n=1}^{N} \frac{Q^n}{w_n} &= \mathbb{Y}_N. \end{aligned} \tag{39}$$

Proof of Correctness.

Step 1 reduces the order to 1 or 2. This step introduces terms of the form Q^{-n}. If $Q = \pm 1$, then $Q^{-n} = Q^n$. Thus, the numerator will remain a w-polynomial if $Q = \pm 1$.

Step 2 guarantees that if there is a denominator, then its first factor will be w_n.

Step 3 ensures that the index of summation begins with 1. The upper limit can be any expression, since N need not be just a variable, but may be any expression.

Step 4 guarantees that there will be no sums (or differences) in the numerator.

Step 5 is justified by the Representation Theorem. Formula (33) comes from [12]. At the end of step 5, there will be no terms of the form w_x in the numerator of any reciprocal sum of order 2.

Step 6 is justified by the Representation Theorem. After step 4, the numerator consists only of a product of terms.

Steps 5 and 6 ensure that these terms only involve w's that cancel with w's in the denominator or are of the form w_{n+1}. Thus, by the time we get to step 7, the only w's left in the numerator are those of the form w_{n+1}. Of course, if the denominator went away, then we are left with a w-polynomial and it is easily summed in step 7.

Step 8 reduces the degree of the variable w_{n+1} to 0 or 1.

Step 9 removed any constants from the numerator.

Step 10 is justified by Theorems 5 and 6. None of the previous steps introduce terms of the form Q^{rn} in the numerator (for $r > 1$). Thus steps 10 and 11 handle all the remaining cases. $\square$

Note. It should be noted that algorithm `WReciprocalSum` also works in the cases where $r < 3$ and $\deg f(n) < 2$ or for any r if $f(n) = Q^{kn}$ where $k = \lfloor r/2 \rfloor$. In step 1, if $f(n) = Q^{kn}$, the Q^{-n} term introduced changes Q^{kn} into $Q^{(k-1)n}$ and the order of the sum decrements by 1 until it reaches 1 or 2. The degree of Q^n will increase only if the degree of $f(n)$ was larger than 1, so if $\deg f(n) < 2$, no terms of the form Q^{cn} are introduced with $c > 1$.

This gives us the following two theorems.

Theorem 11 (The Simplification Theorem for Q-reciprocal sums). Let r be a positive integer and let $k = \lfloor r/2 \rfloor$. Let c_j, $j = 1, 2, \ldots, r$ be distinct integers. Then we can find

$$\sum_{n=1}^{N} \frac{Q^{kn}}{w_{n+c_1} w_{n+c_2} \cdots w_{n+c_r}}$$

in closed form in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$.

Theorem 12 (The Simplification Theorem for Low-Order Reciprocal Sums). Let $f(n)$ be any w-polynomial in the variable n with $\deg f(n) < 2$. Let a and b be distinct integers. Then we can find

$$\sum_{n=1}^{N} \frac{f(n)}{w_{n+a}} \qquad \text{and} \qquad \sum_{n=1}^{N} \frac{f(n)}{w_{n+a} w_{n+b}}$$

in closed form in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$.

7. Some General Formulas.

We have given an algorithm for evaluating certain reciprocal sums. However, in some special cases, simple explicit formulas can be given.

We can take a formula, such as that given by Theorem 4, which involves expressions of the form w_{n+a} and turn it into a valid formula involving expressions of the form $w_{k(n+a)}$ where k is a fixed positive integer. We do this by applying the Dilation Theorem (see [12]) which says we can transform an identity into another identity by replacing all occurrences of w_x by w_{kx} provided that we also replace u_x by u_{kx}/u_k, v_x by v_{kx}, Q by Q^k, P by v_k, and e by eu_k.

Applying the Dilation Theorem to Theorem 4 gives us the following theorem:

Theorem 13. If $a>0$, $k>0$, $u_k\neq 0$, and $u_{ka}\neq 0$, then

$$\begin{aligned}\sum_{n=1}^{N}\frac{Q^{kn}}{w_{kn}w_{k(n+a)}}&=\frac{Q^k}{eu_ku_{ka}}\left[\sum_{n=1}^{a}\frac{w_{k(n-1)}}{w_{kn}}-\sum_{n=1}^{a}\frac{w_{k(N+n-1)}}{w_{k(N+n)}}\right]\\&=\frac{1}{eu_ku_{ka}}\left[\sum_{n=1}^{a}\frac{w_{k(N+n+1)}}{w_{k(N+n)}}-\sum_{n=1}^{a}\frac{w_{k(n+1)}}{w_{kn}}\right].\end{aligned}\tag{40}$$

The last equality comes from the identity

$$\frac{w_{n+1}}{w_n}=\frac{Pw_n-Qw_{n-1}}{w_n}=P-Q\frac{w_{n-1}}{w_n}\tag{41}$$

which when dilated by k gives

$$\frac{w_{k(n+1)}}{w_{kn}}=v_k-Q^k\frac{w_{k(n-1)}}{w_{kn}}.\tag{42}$$

Corollary (letting $a=1$). If $k>0$ and $u_k\neq 0$, then

$$\sum_{n=1}^{N}\frac{Q^{kn}}{w_{kn}w_{k(n+1)}}=\frac{Q^k}{eu_k^2}\left[\frac{w_0}{w_k}-\frac{w_{kN}}{w_{k(N+1)}}\right]=\frac{1}{eu_k^2}\left[\frac{w_{k(N+2)}}{w_{k(N+1)}}-\frac{w_{2k}}{w_k}\right].\tag{43}$$

This agrees with the result given by Lucas [7] in 1878 for the sequences $\langle u_n\rangle$ and $\langle v_n\rangle$. Furthermore, when $k=1$, we get the result found by Kappus [6] which generalized the result of Hillman in [6]. When $w_n=F_n$, this reduces to the results found by Swamy in [14]. When w_n is either the Pell polynomials or the Pell-Lucas polynomials, formula (43) is equivalent to results found by Mahon and Horadam in [8].

In a similar manner, applying the Dilation Theorem to Theorem 5 yields Theorem 14:

Theorem 14. If $a>0$ and $k>0$, then

$$u_{ka}\sum_{n=1}^{N}\frac{Q^{kn}}{w_{kn}w_{k(n+a)}}=u_{kN}\sum_{n=1}^{a}\frac{Q^{kn}}{w_{kn}w_{k(n+N)}}.\tag{44}$$

Theorem 15. If $a>0$, $b>0$, $k>0$, and $u_{ka}\neq 0$, then

$$\sum_{n=1}^{N}\frac{Q^{kn}}{w_{kn+b}w_{k(n+a)+b}}=\frac{u_{kN}}{u_{ka}}\sum_{n=1}^{a}\frac{Q^{kn}}{w_{kn+b}w_{k(n+N)+b}}.\tag{45}$$

Proof: Apply the Translation Theorem (see [12]) to Theorem 6 to convert the sequence $\langle w_n \rangle$ into the sequence $\langle w_{n+b} \rangle$. $\square$

If $P = x$ and $Q = -1$, Theorems 5, 6, and 14 give results about partial sums of Fibonacci polynomials that were found by Bergum and Hoggatt [2].

Corollary (letting $a = 1$). If $b > 0$, $k > 0$, and $u_k \neq 0$, then

$$\sum_{n=1}^{N} \frac{Q^{k(n-1)}}{w_{kn+b}w_{k(n+1)+b}} = \frac{u_{kN}}{u_k w_{k+b} w_{k(N+1)+b}}. \tag{46}$$

This is equivalent (with $b = a - k$) to the results found by Popov [10] for the sequences $\langle u_n \rangle$ and $\langle v_n \rangle$.

Theorem 16. If $a < b$, $k > 0$, and $u_{k(b-a)} \neq 0$, then

$$\sum_{n=1}^{N} \frac{Q^{kn}}{w_{k(n+a)}w_{k(n+b)}} = \frac{u_{kN}}{u_{k(b-a)}} \sum_{n=1}^{b-a} \frac{Q^{kn}}{w_{k(n+a)}w_{k(n+N+a)}}. \tag{47}$$

Proof: Apply the Translation Theorem (see [12]) to change the sequence $\langle w_m \rangle$ in Theorem 6 into the sequence $\langle w_{m+ka} \rangle$. Then let $c = b - a$. $\square$

Theorem 17. If $k > 0$, $c > 0$, and $u_{kc} \neq 0$, then

$$\sum_{n=1}^{N} \frac{1}{w_{kn}w_{k(n+c)}} = \frac{u_k}{u_{kc}} \sum_{i=0}^{c-1} Q^{ki}(\mathbb{W}_{k,N+i} - \mathbb{W}_{k,i}). \tag{48}$$

where

$$\mathbb{W}_{k,N} = \sum_{n=1}^{N} \frac{1}{w_{kn}w_{k(n+1)}}.$$

Proof: Apply the Dilation Theorem to Theorem 5. $\square$

Applying the Translation Theorem to Theorem 15 gives us the following result.

Theorem 18. If $a < b$ and $u_{k(b-a)} \neq 0$, then

$$\sum_{n=1}^{N} \frac{1}{w_{k(n+a)}w_{k(n+b)}} = \frac{u_k}{u_{k(b-a)}} \sum_{i=0}^{b-a-1} Q^{ki}(\mathbb{W}_{k,a+N+i} - \mathbb{W}_{k,a+i}). \tag{49}$$

8. Special Cases.

Although unit reciprocal sums of order 2 cannot in general be evaluated in closed form (without involving terms of the form $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, or $\mathbb{Y}_N$), a closed form can be found for some important special cases (such as when $Q = \pm 1$).

Theorem 19. If $Q = -1$, then

$$\sum_{n=1}^{N} \frac{1}{w_{n+a}w_{n+a+2}} = \frac{1}{P}\left[\frac{1}{w_{a+1}w_{a+2}} - \frac{1}{w_{N+a+1}w_{N+a+2}}\right]. \tag{50}$$

Proof: Since $Q = -1$, we have $w_{m+2} = Pw_{m+1} + w_m$. Thus,

$$\begin{aligned}\frac{P}{w_{n+a}w_{n+a+2}} &= \frac{Pw_{n+a+1}}{w_{n+a}w_{n+a+1}w_{n+a+2}} = \frac{w_{n+a+2} - w_{n+a}}{w_{n+a}w_{n+a+1}w_{n+a+2}} \\ &= \frac{1}{w_{n+a}w_{n+a+1}} - \frac{1}{w_{n+a+1}w_{n+a+2}}.\end{aligned}$$

Summing from 1 to N gives the desired result since the right-hand side telescopes. □

Lemma. If $Q = 1$, then

$$c_k = \frac{r_1^k w_{k(n+1)} - w_{kn}}{r_1^{k(n+1)}} \tag{51}$$

is independent of n. In particular, $c_k = (w_1 - w_0)r_2 u_k$.

Proof: Since $Q = 1$, we have $r_1 r_2 = 1$. The Binet form for w_n is known to be

$$w_n = Ar_1^n + Br_2^n$$

where $A = \frac{w_1 - w_0 r_2}{r_1 - r_2}$ and $B = \frac{w_0 r_1 - w_1}{r_1 - r_2}$. Then

$$\begin{aligned}c_k r_1^{k(n+1)} &= r_1^k w_{k(n+1)} - w_{kn} = r_1^k\left[Ar_1^{k(n+1)} + Br_2^{k(n+1)}\right] - \left[Ar_1^{kn} + Br_2^{kn}\right] \\ &= Ar_1^{kn+2k} - Ar_1^{kn} \\ &= Ar_1^{kn}[r_1^{2k} - 1] \\ &= Ar_1^{kn}[r_1^{2k} - (r_1 r_2)^k] \\ &= Ar_1^{k(n+1)}[r_1^k - r_2^k] \\ &= Ar_1^{k(n+1)}(r_1 - r_2)u_k.\end{aligned}$$

Therefore $c_k = A(r_1 - r_2)u_k = (w_1 - w_0 r_2)u_k$. □

Theorem 20. If $Q = 1$, $k \neq 0$, and $u_k \neq 0$, then

$$\sum_{n=1}^{N} \frac{1}{w_{kn}w_{k(n+1)}} = \frac{1}{(w_1 - w_0 r_2) r_1^k u_k} \left[\frac{1}{w_k} - \frac{1}{r_1^{kN} w_{k(N+1)}} \right]. \tag{52}$$

Proof: Using the Lemma, we have

$$\frac{1}{r_1^{kn} w_{kn}} - \frac{1}{r_1^{k(n+1)} w_{k(n+1)}} = \frac{r_1^k w_{k(n+1)} - w_{kn}}{r_1^{k(n+1)} w_{kn} w_{k(n+1)}} = \frac{(w_1 - w_0 r_2) u_k}{w_{kn} w_{k(n+1)}}.$$

Summing as n goes from 1 to N, we find that the left side telescopes and we reach the stated result. $\square$

This theorem generalizes the results found by Melham and Shannon [9]. The idea for the proof comes from that paper. Alternatively, we could let $Q = 1$ in formula (43).

Corollary (letting $k = 1$). If $Q = 1$, then

$$\sum_{n=1}^{N} \frac{1}{w_n w_{n+1}} = \frac{1}{(w_1 - w_0 r_2) r_1} \left[\frac{1}{w_1} - \frac{1}{r_1^N w_{N+1}} \right]. \tag{53}$$

9. Open Problems.

Query 1. Is there a simple closed form for any of the quantities $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, or $\mathbb{Y}_N$?

Query 2. Is there a simple algebraic relationship between any of the quantities $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$?

Query 3. Can $\displaystyle\sum_{n=1}^{N} \frac{1}{w_{2n} w_{2n+1}}$ be expressed in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$?

Query 4. Can $\displaystyle\sum_{n=1}^{N} \frac{1}{w_n^2}$ be expressed in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$?

Query 5. Can $\displaystyle\sum_{n=1}^{N} \frac{1}{w_n w_{n+1} w_{n+2}}$ be expressed in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, and $\mathbb{Y}_N$?

Query 6. Can $\displaystyle\sum_{n=1}^{N} \frac{1}{w_{n+a}w_{n+b}w_{n+c}}$ be expressed in terms of $\mathbb{U}_N$, $\mathbb{V}_N$, $\mathbb{W}_N$, $\mathbb{X}_N$, $\mathbb{Y}_N$, a, b, and c?

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July 7, 2018 Addendum

Equation (12) is incorrect. The correct result is the following.

If $a > 1$, then

$$\sum_{n=1}^{N} \frac{Q^n}{w_{n+a}} = \frac{1}{Q^a} \left[\mathbb{Y}_{N+a} - \mathbb{Y}_a \right]. \tag{12$'$}$$

Proof: Letting $k = n + a$ gives

$$\begin{aligned}
\sum_{n=1}^{N} \frac{Q^n}{w_{n+a}} &= \sum_{k=a+1}^{a+N} \frac{Q^{k-a}}{w_k} \\
&= Q^{-a} \sum_{k=a+1}^{a+N} \frac{Q^k}{w_k} \\
&= Q^{-a} \left[\sum_{k=1}^{a+N} \frac{Q^k}{w_k} - \sum_{k=1}^{a} \frac{Q^k}{w_k} \right] \\
&= \frac{1}{Q^a} \left[\mathbb{Y}_{N+a} - \mathbb{Y}_a \right].
\end{aligned}$$