The Seven Circles Theorem

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Start with a circle. Any circle. Draw six more circles inside it, each internally tangent to the original circle and tangent to each other in pairs. Let A, B, C, D, E, and F be the consecutive points of tangency of the small circles with the outer circle. We wind up with a set of seven circles as shown in Figure 1. The *Seven Circles Theorem* says that no matter what sizes we pick for the seven circles (subject only to certain order and tangency constraints), it will turn out that the lines AD, BE, and CF will meet in a point.

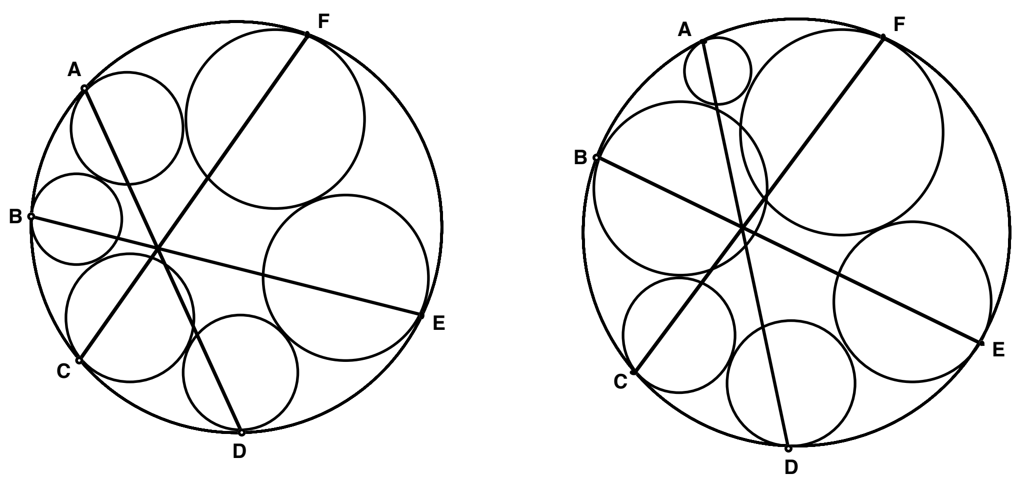

Figure 1

This remarkable theorem is less than fifteen years old. It shows that there are many beautiful relationships involving only lines and circles still waiting to be discovered. Evelyn, Money-Coutts, and Tyrrell [6] first published this theorem in 1974. Since then, other proofs have appeared (see [5]). The purpose of this article is to give a simple proof of this theorem using only elementary geometry.

Since we wish to prove that three lines *concur* (meet in a point), we start by reviewing what is known about three concurrent lines. Various facts about three concurrent lines in a triangle were known to early geometers (like Heron of Alexandria and Archimedes). They knew that the medians concur and the altitudes concur ([2], pp. 297–298). However, it was not until 1678 that Giovanni Ceva gave a definitive treatment about such lines. For that reason, a line from a vertex of a triangle to a point on the opposite side is called a *cevian*. Here is a simplified version of Ceva's Theorem.

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Ceva's Theorem. Let D, E, and F be points on sides BC, CA, AB, respectively, of triangle ABC. Then cevians AD, BE, CF concur if and only if $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

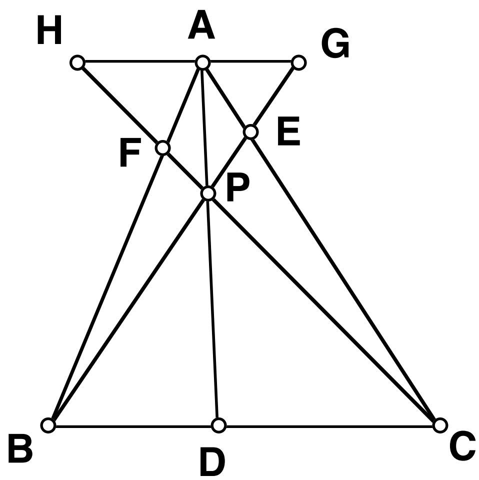

Figure 2

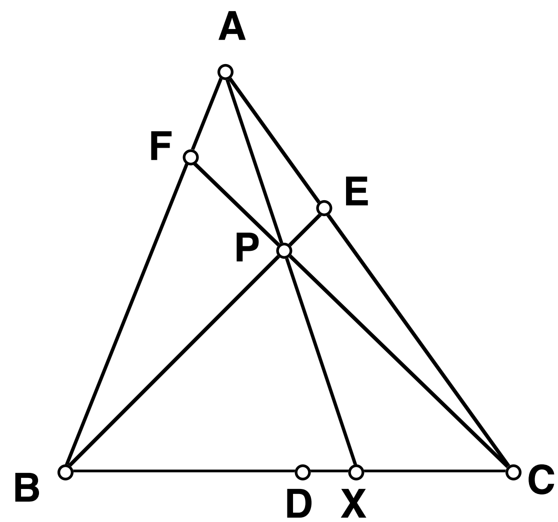

Figure 3

Proof. (i) Suppose AD, BE, CF meet at a point P. Extend BE and CF until they meet the line through A that is parallel to BC at points G and H respectively (see Figure 2). From similar triangles, we get the proportions:

$$DC/HA = PD/AP$$
$$AG/BD = AP/PD$$
$$AE/EC = AG/BC$$

and
$$BF/FA = BC/HA.$$

Multiplying these together gives us the desired result.

(ii) Conversely, suppose

$$AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

Let BE meet CF at P and let AP meet BC at X. Then by part (i), we have

$$AF \cdot BX \cdot CE = FB \cdot XC \cdot EA.$$

Dividing these two results gives

$$\frac{BD}{DC} = \frac{BX}{XC}.$$

If X does not coincide with D, then without loss of generality, assume X lies on segment DC (see Figure 3). Then $BD < BX$ and $DC > XC$. Consequently, $BD/DC < BX/XC$, a contradiction. Thus X coincides with D.

Remark. If we are a little more careful about signs and use directed segments, Ceva's Theorem can be generalized to work for any points D, E, F on the sides of the triangle or on the extensions of these sides (see [2]). However, we will not need this extended result here.

Before we can approach the Seven Circles Theorem, we must know something about when three chords of a circle concur. Coxeter [3] gives a criterion that we will call *Ceva's Theorem for Chords.* Both the statement and proof are remarkably analogous to Ceva's Theorem.

Ceva's Theorem for Chords. Let A, B, C, D, E, and F be six consecutive points around the circumference of a circle. Then chords AD, BE, CF concur if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

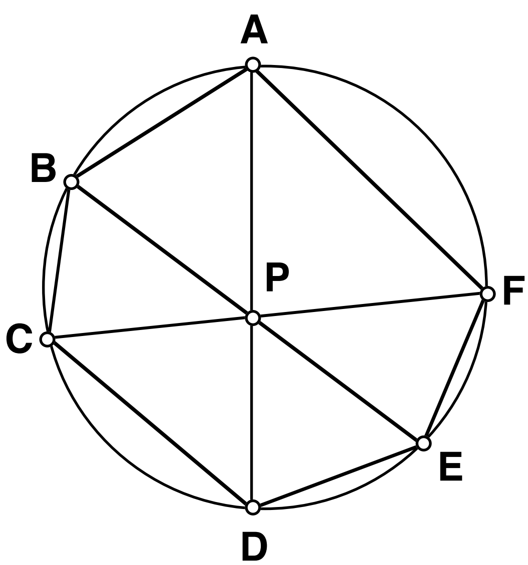

Figure 4

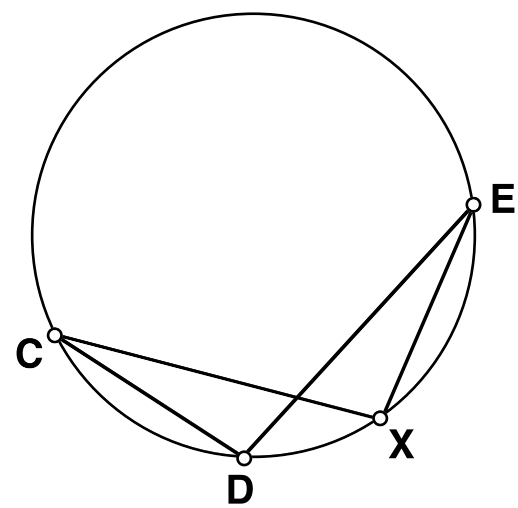

Figure 5

Proof. (i) Suppose AD, BE, CF meet at a point P (see Figure 4). From similar triangles, we get the proportions:

$$AB/DE = PA/PE$$
$$EF/BC = PF/PB$$
$$CD/FA = PC/PA$$

and

$$PC/PE = PB/PF.$$

Multiplying these together gives us the desired result.

(ii) Conversely, suppose

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA. \tag{1}$$

Of the three arcs, $A\overset{\frown}{B}C$, $C\overset{\frown}{D}E$, $E\overset{\frown}{F}A$, at least one must be smaller than a semicircle. Without loss of generality, assume arc $C\overset{\frown}{D}E$ is smaller than a semicircle. Let BE meet CF at point P and let AP meet the circle again at point X (which must lie on arc $C\overset{\frown}{D}E$). By part (i), we have

$$AB \cdot CX \cdot EF = BC \cdot XE \cdot FA.$$

This combined with (1) gives

$$\frac{CD}{DE} = \frac{CX}{XE}.$$

If X does not coincide with D, then without loss of generality, assume X lies on arc $\overset{\frown}{DE}$ (see Figure 5). Then $CD < CX$ and $DE > XE$. Consequently, $CD/DE < CX/XE$, a contradiction. Thus X must coincide with D.

Before proceeding to the Seven Circles Theorem, we need one preliminary result.

Lemma. Let two externally tangent circles, P and Q, be internally tangent to circle C at points A and B respectively. If the radii of circles C, P, and Q are R, p, and q, respectively, then $AB^2/4R^2 = (p/(R-p)) \cdot (q/(R-q))$.

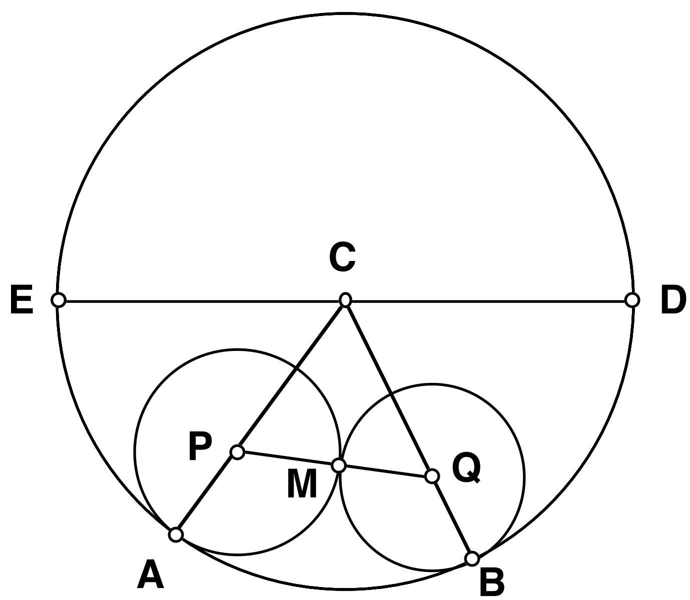

Figure 6

Proof. Let circles P and Q be tangent at point M. Extend AM and BM to meet circle C again at points D and E respectively. Identify the names of circles with their centers. Draw CD and CE. (See Figure 6.) Draw PQ which must pass through M.

$CA = CD$ implies $\angle CAD = \angle CDA$. $PA = PM$ implies $\angle PAM = \angle PMA$. Therefore, $\angle PMA = \angle CDA$ and $CD \parallel PM$. Similarly, $CE \parallel QM$. But PMQ is a straight line, so therefore DCE is a straight line also. Note also that $\angle EBA = \angle EDA$ (since both measure half of arc $\overset{\frown}{EA}$). We thus have three pairs of similar triangles: $\triangle MDE \sim \triangle MBA$, $\triangle BMQ \sim \triangle BEC$, and $\triangle AMP \sim \triangle ADC$. Then

$$AB/DE = MA/ME = MB/MD.$$

Since $DE = 2R$, we have

$$\frac{AB}{2R} \cdot \frac{AB}{2R} = \frac{MA}{ME} \cdot \frac{MB}{MD} = \frac{MA}{MD} \cdot \frac{MB}{ME} = \frac{PA}{CP} \cdot \frac{QB}{CQ} = \frac{p}{R-p} \cdot \frac{q}{R-q}.$$

We are now ready to prove our main result.

The Seven Circles Theorem. Let A_0, A_1, A_2, A_3, A_4, A_5 be six consecutive points around the circumference of a circle O. Suppose circles can be drawn internally tangent to circle O at these six points so that they are also externally tangent to each other in pairs (that is, the circle at A_i is tangent to the circle at A_{i-1} and the circle at A_{i+1}, where subscripts are reduced modulo 6). (See Figure 7.) Then segments A_0A_3, A_1A_4, and A_2A_5 concur.

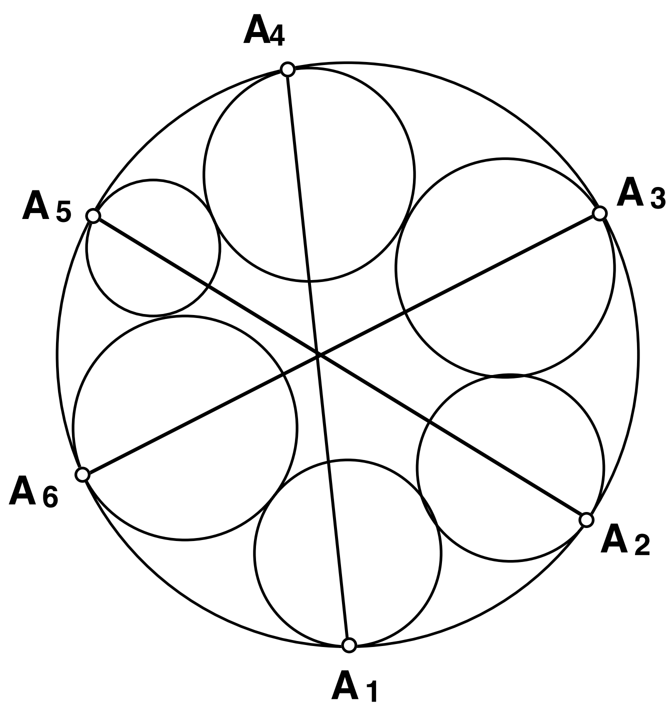

Figure 7

Proof. Let the radius of circle O be R and let the radius of the circle at A_i be r_i. Let us express A_iA_{i+1} in terms of r_i and r_{i+1}. By the lemma, we have

$$A_iA_{i+1} = 2Rf(r_i)f(r_{i+1})$$

where $f(r) = \sqrt{r/(R-r)}$ and the subscripts are reduced modulo 6. Thus

$$A_0A_1 \cdot A_2A_3 \cdot A_4A_5 = 8R^3 f(r_0)f(r_1)f(r_2)f(r_3)f(r_4)f(r_5) = A_1A_2 \cdot A_3A_4 \cdot A_5A_0.$$

So by Ceva's Theorem for Chords, A_0A_3, A_1A_4, and A_2A_5 must concur.

The Seven Circles Theorem is true for more general configurations than the one described above. For example, Figure 8 shows the case where the six circles are externally tangent to the original circle rather than being internally tangent.

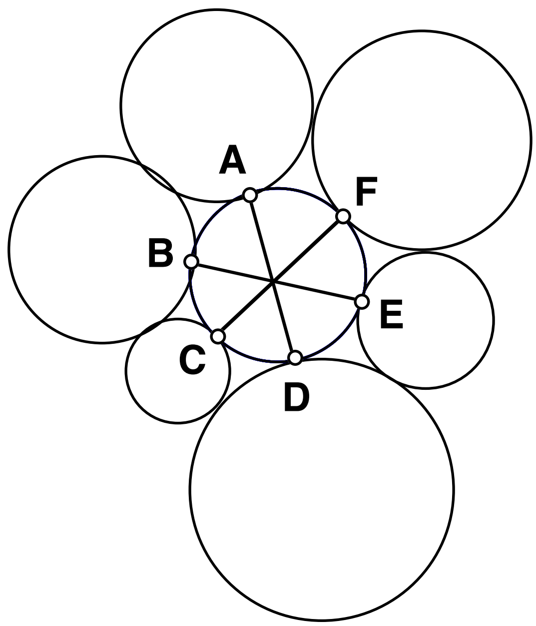

Figure 8

This case can be proved in a manner similar to the previous proof. Using Figure 9, we can derive the formula

$$AB^2/4R^2 = (p/(R+p)) \cdot (q/(R+q))$$

whose proof is analogous to the proof of the preceding lemma. Here, circles P and Q are externally tangent to circle C. R, p, and q denote the radii of circles C, P, and Q respectively. It then becomes clear that

$$A_0A_1 \cdot A_2A_3 \cdot A_4A_5 = 8R^3 g(r_0)g(r_1)g(r_2)g(r_3)g(r_4)g(r_5) = A_1A_2 \cdot A_3A_4 \cdot A_5A_0$$

where $g(r) = \sqrt{r/(R+r)}$.

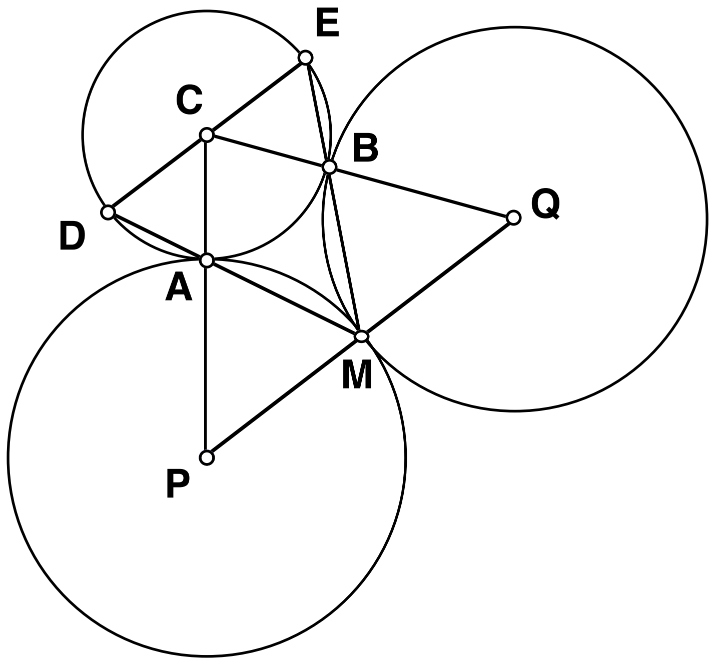

Figure 9

In fact, the Seven Circles Theorem is even more general than this. The six points of tangency need not occur successively along the circumference of the original circle. Two such cases are shown in Figure 10. A proof for the general configuration can be found in [6].

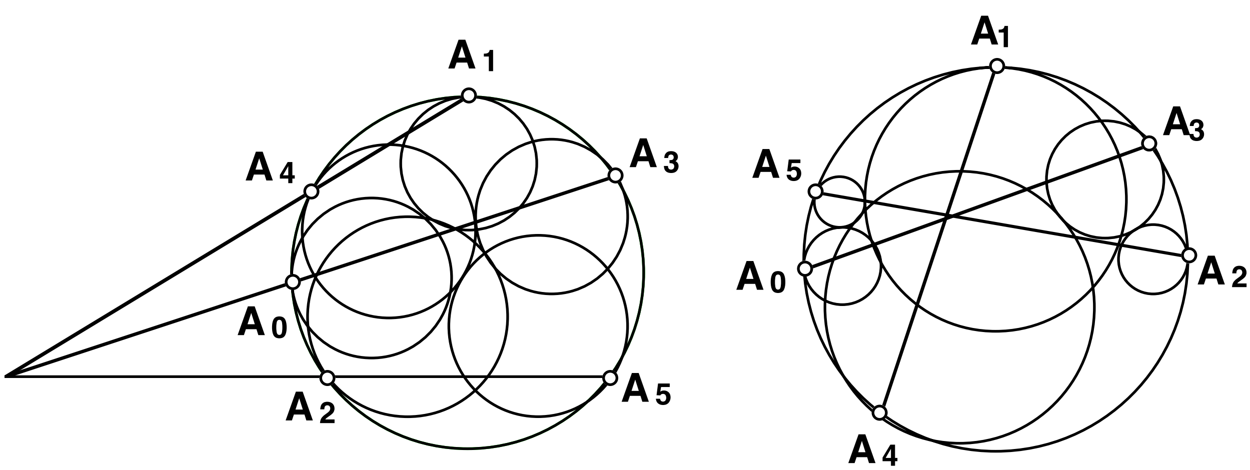

Figure 10

An interesting subtlety occurs when trying to formulate the theorem for the most general configuration. After starting with an initial circle, C, and drawing five circles, A_1, A_2, A_3, A_4, and A_5 tangent to C and tangent to themselves in succession, it becomes necessary to draw a sixth circle tangent to C, A_5, and A_1. However, in general, this can be done in two ways (see Figure 11). Of the two choices, one satisfies the conclusion of the Seven Circles Theorem and the other doesn't. In this sense, the Seven Circles Theorem may be thought to hold only 50% of the time.

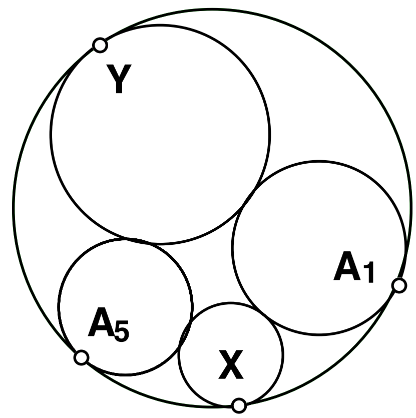

Figure 11

Exercises. We conclude with a few exercises to allow readers to try some related problems.

1. A circle is inscribed in triangle ABC. The points of contact with sides BC, CA, and AB are D, E, and F respectively (see Figure 12). Prove that AD, BE, and CF concur. (The point of concurrence is known as the *Gergonne point* of the triangle; see [2], page 160.) Show further that the conclusion still holds if the circle is replaced by an ellipse.

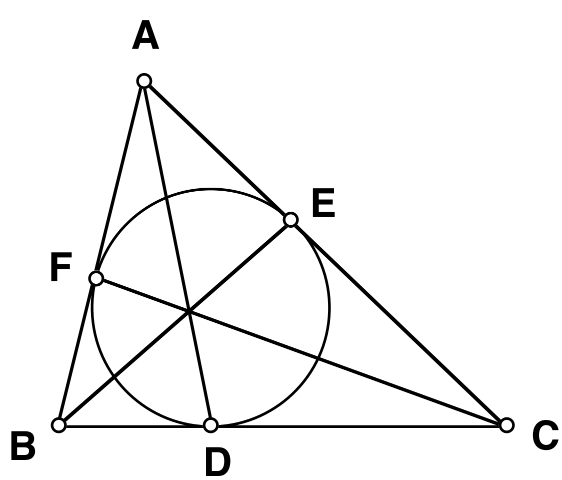

Figure 12

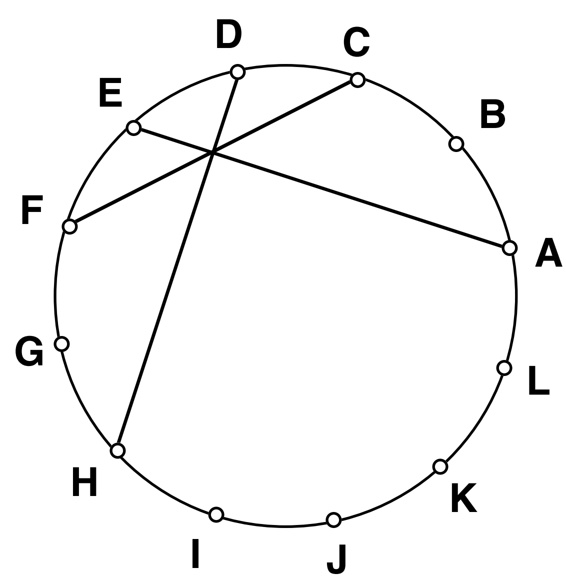

Figure 13

2. Let $ABCDEFGHIJKL$ be a regular dodecagon (see Figure 13). Prove that diagonals AE, CF, and DH concur. (For a proof see [10].)

3. Three circles are situated as shown in Figure 14 so that each meets the others in two points. Prove that AD, BE, CF concur and that $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$. (This result is due to Haruki, see [9].)

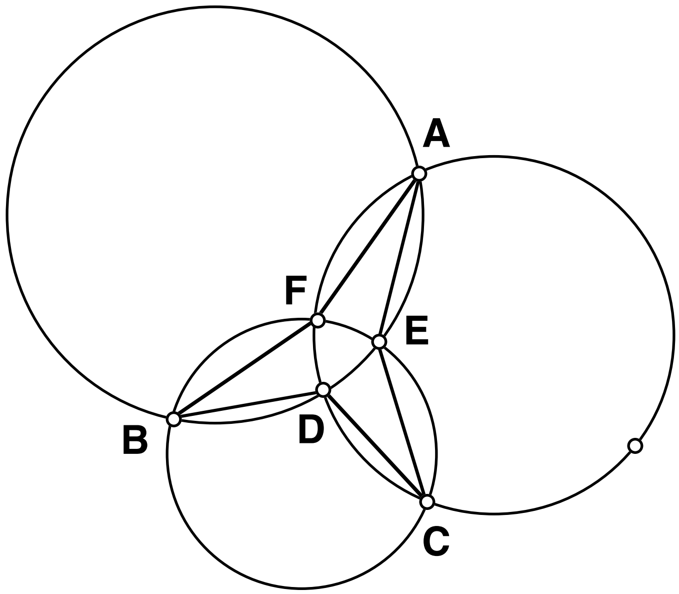

Figure 14

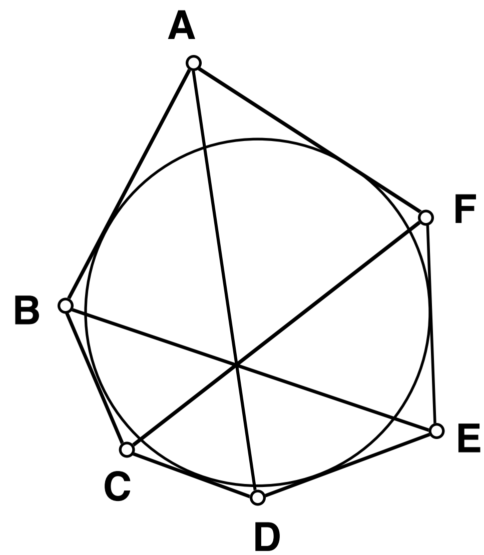

Figure 15

4. Let $ABCDEF$ be a hexagon circumscribed about a circle, as in Figure 15. Prove that AD, BE, CF concur. (This is a special case of Brianchon's Theorem, see [4], p. 77.)
5. Let P be a point inside pentagon $ABCDE$ such that lines AP, BP, CP, DP, EP meet the opposite sides at points F, G, H, I, and J as shown in Figure 16. Prove that $AI \cdot BJ \cdot CF \cdot DG \cdot EH = BI \cdot CJ \cdot DF \cdot EG \cdot AH$. (See [8], page 67.)

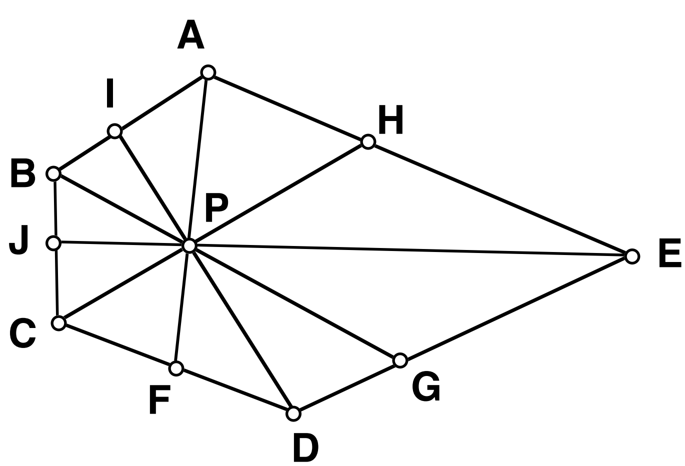

Figure 16

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