Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 3 (2019), pp.X-X
Received XX XXX XXX. Published on-line XX XXX XXX
web: http://www.sangaku-journal.eu/

Relationships Between Six Circumcircles

Stanley Rabinowitz
545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com/

Abstract. If P is a point inside $\triangle ABC$, then the cevians through P divide $\triangle ABC$ into small triangles. We give theorems about the relationship between the radii of the circumcircles of these triangles.

Keywords. Euclidean geometry, triangle geometry, excircles, exradii, cevians.

Mathematics Subject Classification (2010). 51M04.

Let P be any point inside a triangle ABC. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1.

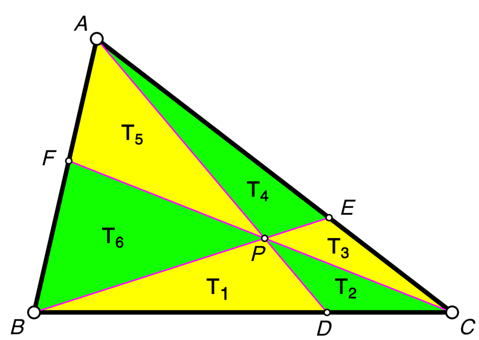

Figure 1. numbering of the six triangles

The relationships between the radii of the circles inscribed in these triangles was investigated in [6]. The relationships between the radii of certain excircles associated with these triangles was investigated in [5]. In this paper, we will investigate the relationships between the radii of the circles cicumscribed about these triangles.

We will make use of The Extended Law of Sines which states that if a, b, and c are the lengths of the sides of a triangle opposite angles A, B, and C, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of $\triangle ABC$.

Theorem 1. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let R_i be the circumradius of T_i. Then $R_1R_3R_5 = R_2R_4R_6$.*

Proof. By The Extended Law of Sines in $\triangle PBD$, we have

$$R_1 = \frac{BD}{2\sin\angle BPD},$$

with similar expressions for the other R_i. Thus,

$$R_1R_3R_5 = \frac{BD}{2\sin\angle BPD} \cdot \frac{CE}{2\sin\angle CPE} \cdot \frac{AF}{2\sin\angle APF}$$

and

$$R_2R_4R_6 = \frac{DC}{2\sin\angle DPC} \cdot \frac{EA}{2\sin\angle EPA} \cdot \frac{FB}{2\sin\angle FPB}.$$

But $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$ by Ceva's Theorem. Also, angles BPD and EPA are vertical angles, so they are congruent and their sines are equal. Similarly, $\sin\angle CPE = \sin\angle FPB$ and $\sin\angle APF = \sin\angle DPC$. Therefore, we conclude that $R_1R_3R_5 = R_2R_4R_6$. □

We have some additional results for specific locations of point P.

Theorem 2. *Let O be the circumcenter of $\triangle ABC$ and assume that O lies inside $\triangle ABC$. The cevians through O divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let R_i be the circumradius of T_i. Then $R_1 = R_2$, $R_3 = R_4$, and $R_5 = R_6$.*

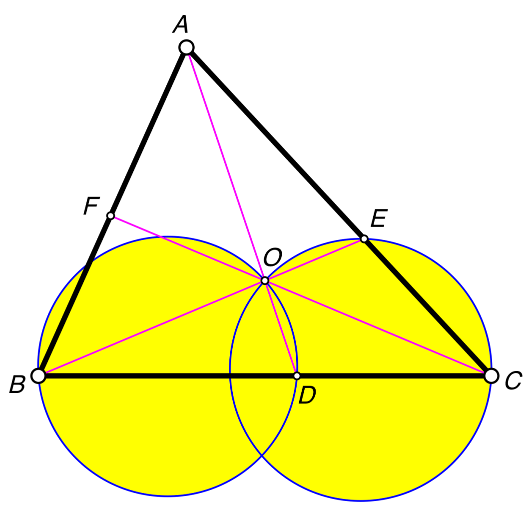

FIGURE 2. Circumcenter: $R_1 = R_2$

Proof. By symmetry, it suffices to show that $R_1 = R_2$ (Figure 2). Since O is the circumcenter of $\triangle ABC$, $OB = OC$. Angles ODB and ODC are supplementary, so their sines are equal. Thus, by The Extended Law of Sines, we have

$$R_1 = \frac{OB}{2\sin\angle ODB} = \frac{OC}{2\sin\angle ODC} = R_2$$

as required. □

Theorem 3. *Let N be the Nagel Point of $\triangle ABC$. The cevians through N divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let R_i be the circumradius of T_i. Then $R_1 = R_4$, $R_2 = R_5$, and $R_3 = R_6$.*

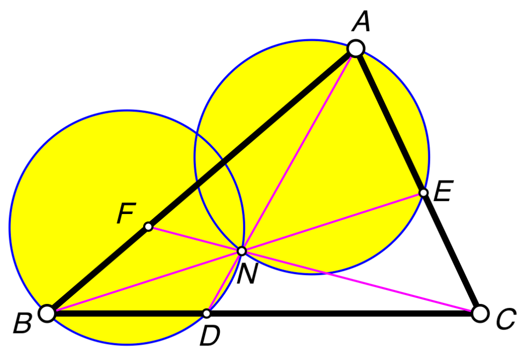

FIGURE 3. Nagel Point: $R_1 = R_4$

Note: The Nagel Point of a triangle is the point of concurrence of AD, BE, and CF, where D, E, and F are the points where the excircles of $\triangle ABC$ touch the sides BC, CA, and AB, respectively [1, p. 160]. The Nagel point is usually denoted Na, but here we will name it N, for simplicity.

Proof. First note that by symmetry, it suffices to show that $R_1 = R_4$ (Figure 3). If $BC = a$, $CA = b$, $AB = c$, and $s = (a + b + c)/2$, then it is known that $BD = AE = s - c$ [1, p. 88]. Thus, by The Extended Law of Sines and the fact that $\angle BND = \angle ENA$, we have

$$R_1 = \frac{BD}{2\sin\angle BND} = \frac{AE}{2\sin\angle ENA} = R_4$$

as required. □

Theorem 4. *Let H be the orthocenter of $\triangle ABC$. The cevians through H divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let C_i be the circumcircle of T_i. Let R_i be the radius of C_i. Then $R_1 = R_6$, $R_2 = R_3$, and $R_4 = R_5$.*

Proof. By symmetry, it suffices to show that $R_1 = R_4$ (Figure 4), i.e., that C_1 and C_6 coincide. Since $\angle BDH + \angle HFB = 180°$, quadrilateral $BDHF$ is cyclic. Thus, the circle through points B, D, and H is the same as the circle through points B, F, and H. □

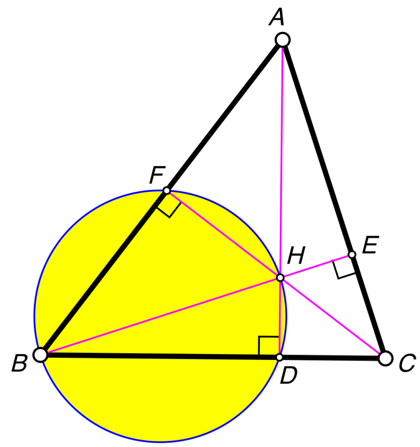

FIGURE 4. Orthocenter: $R_1 = R_6$

We also have some interesting result concerning the centers of the six circumcircles. We will use the notation [XYZ] to denote the area of $\triangle XYZ$.

Theorem 5. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then $[O_1O_3O_5] = [O_2O_4O_6]$ (Figure 5).*

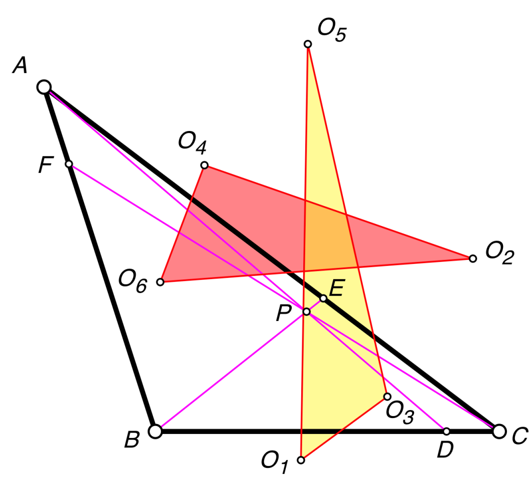

FIGURE 5. Two triangles have same area

The following proof is due to Dubrovsky [2].

Proof. Since O_1 is the circumcenter of $\triangle BPD$, it must lie on the perpendicular bisector of BP. The same remark holds true for O_6. Therefore, $O_1O_6 \perp BP$. In the same way, $O_6O_5 \perp PF$, $O_5O_4 \perp AP$, $O_4O_3 \perp PE$, $O_3O_2 \perp CP$, and $O_2O_1 \perp PD$. Hence $O_1O_6 \parallel O_3O_4$, $O_6O_5 \parallel O_2O_3$, and $O_5O_4 \parallel O_1O_2$. Therefore, hexagon $O_1O_2O_3O_4O_5O_6$ has its opposite sides parallel. But it is known [3] that if $ABCDEF$ is a hexagon with its opposite sides parallel, then $[ACE] = [BDF]$. Thus $[O_1O_3O_5] = [O_2O_4O_6]$. □

Theorem 6. *Let M be the centroid of $\triangle ABC$. The medians through M divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then $O_1O_4 = O_2O_5 = O_3O_6$. (Figure 6).*

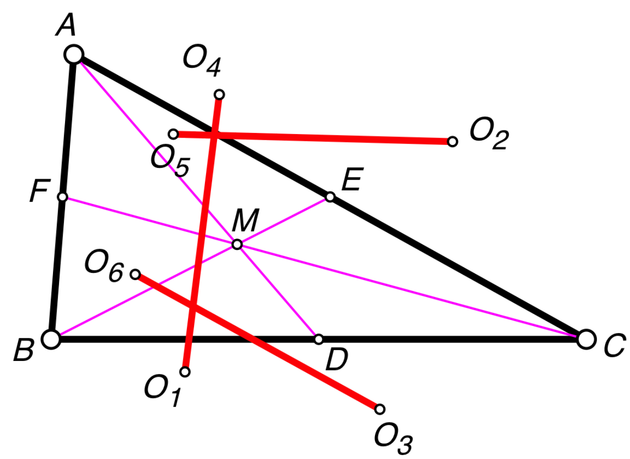

FIGURE 6. Red segments are congruent

Proof. This follows from Proposition 4 of [4]. □

The following two results come from [4].

Theorem 7. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then the points O_i lie on a circle if and only if either P is the centroid of $\triangle ABC$ (Figure 7) or P is the orthocenter of $\triangle ABC$ (in which case $O_6 = O_1$, $O_2 = O_3$, and $O_4 = O_5$).*

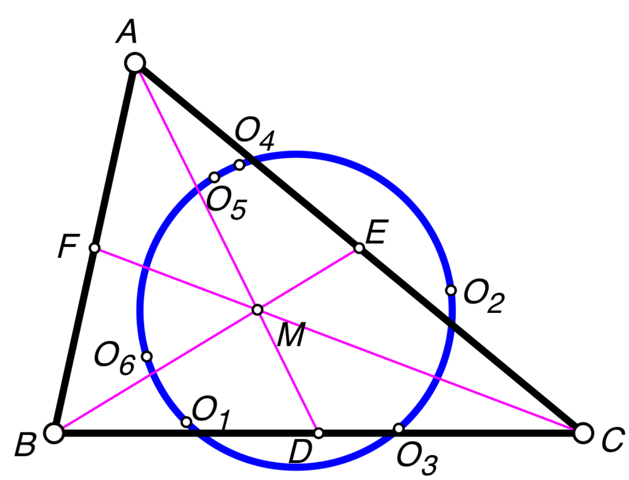

FIGURE 7. O_i lie on a circle when $P = M$

Theorem 8. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then the points O_i lie on a conic (Figure 8).*

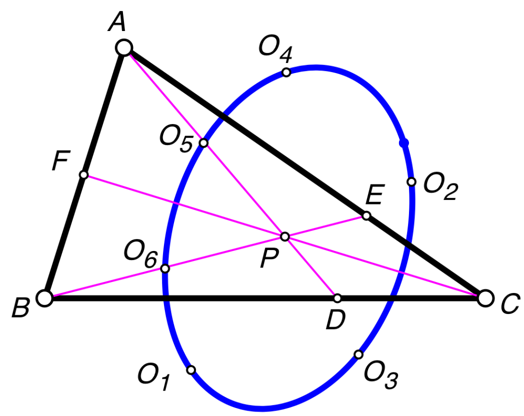

FIGURE 8. O_i lie on a conic

Proof. Since $O_1O_6 \parallel O_3O_4$, $O_6O_5 \parallel O_2O_3$, and $O_5O_4 \parallel O_1O_2$, the result follows from the converse of Pascal's Theorem. □

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