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Relationships Between Six Circles Associated with a Triangle

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Abstract. If P is a point inside $\triangle ABC$, then the cevians through P divide $\triangle ABC$ into six small triangles. We give theorems about the relationship between the radii of the circles inscribed in these triangles and the lengths of the segments formed along the sides of the triangle.

Keywords. Euclidean geometry, triangle geometry, incircles, cevians.

Mathematics Subject Classification (2010). 51M04.

Let P be any point inside a triangle ABC. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1.

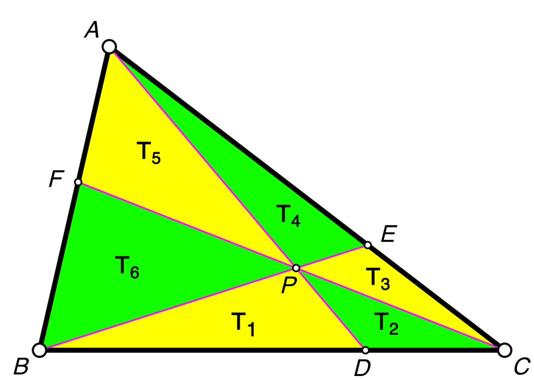

Figure 1. numbering of the six triangles

Let the radius of the circle inscribed in triangle T_i be r_i. The relationships between the r_i was investigated in [1]. For special points, P, such as the centroid, the circumcenter, and the orthocenter, formulas were found relating the r_i, independent of the shape of the triangle. No such formula was found when P is an arbitrary point inside the triangle.

In this paper, we will investigate the relationships between the r_i and the lengths of the segments formed by points D, E, and F on the sides of the triangle.

We will use the following notation throughout this paper. Let K_i be the area of T_i. Let s_i be the semiperimeter of T_i. Let O_i be the incenter of T_i. Let $AB = c$, $BC = a$, and $CA = b$. The cevians will be AD, BC, and CE, and the twelve segments formed by them with each other and the sides of the triangle have lengths as shown in Figure 2.

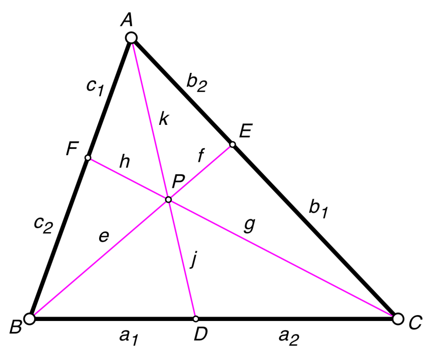

FIGURE 2. lengths of the twelve segments

The purpose of this paper is to give a simple formula connecting the r_i and a_1, a_2, b_1, b_2, c_1, and c_2.

We start with several lemmas. The following result was stated by van Aubel in 1882 [3] and is often called Van Aubel's Theorem for Triangles [4].

Lemma 1 (Van Aubel's Theorem for Triangles)**.** *Let P be any point inside $\triangle ABC$ and let the cevians through P be AD, BE, and CF (Figure 3). Then*

$$\frac{AF}{FB} + \frac{AE}{EC} = \frac{AP}{PD}.$$

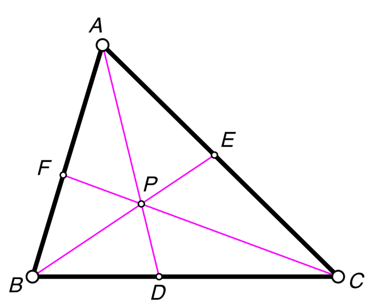

FIGURE 3.

An immediate consequence of Van Aubel's Theorem for Triangles is the following lemma.

Lemma 2. *Using the notation of Figure 2, we have the following equations.*

$$e = f\left(\frac{c_2}{c_1} + \frac{a_1}{a_2}\right), \quad g = h\left(\frac{b_1}{b_2} + \frac{a_2}{a_1}\right), \quad k = j\left(\frac{c_1}{c_2} + \frac{b_2}{b_1}\right).$$

Lemma 3. *The K_i can be expressed as multiples of K_1 using the lengths shown in Figure 2. In particular,*

$$K_2 = K_1\left(\frac{a_2}{a_1}\right), \quad K_3 = K_1\left(\frac{(a_1+a_2)f}{a_1 e}\right), \quad K_4 = K_1\left(\frac{(a_1+a_2)b_2 f}{a_1 b_1 e}\right),$$
$$K_5 = K_1\left(\frac{(a_1+a_2)(b_1+b_2)fh}{a_1 b_1 e g}\right), \quad K_6 = K_1\left(\frac{(a_1+a_2)(b_1+b_2)c_2 fh}{a_1 b_1 c_1 e g}\right).$$

Proof. If two triangles have the same altitude, then their areas are proportional to their bases. This gives the following proportions.

$$\frac{K_2}{K_1} = \frac{a_2}{a_1}, \quad \frac{K_3}{K_1+K_2} = \frac{f}{e}, \quad \frac{K_4}{K_3} = \frac{b_2}{b_1}, \quad \frac{K_5}{K_3+K_4} = \frac{h}{g}, \quad \frac{K_6}{K_5} = \frac{c_2}{c_1}.$$

Some algebraic manipulation then gives us the desired equations. □

Theorem 1. *Let P be any point inside $\triangle ABC$. Then*

$$\frac{a_2c_2}{r_1} + \frac{a_2c_1}{r_3} + \frac{a_1c_1}{r_5} = \frac{a_2c_2}{r_2} + \frac{a_2c_1}{r_4} + \frac{a_1c_1}{r_6}.$$

Proof. Let

$$S = a_2c_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + a_2c_1\left(\frac{1}{r_3} - \frac{1}{r_4}\right) + a_1c_1\left(\frac{1}{r_5} - \frac{1}{r_6}\right).$$

We want to show that $S = 0$. Use the formula for the inradius of a triangle to replace r_i with K_i/s_i. Then replace each s_i with the semiperimeter of triangle T_i as found in Figure 2.

$$S = a_2c_2\left(\frac{a_1+e+j}{2K_1} - \frac{a_2+g+j}{2K2}\right) + a_2c_1\left(\frac{b_1+f+g}{2K_3} - \frac{b_2+f+k}{2K_4}\right)$$
$$+ a_1c_1\left(\frac{c_1+h+k}{2K_5} - \frac{c_2+e+h}{2K_6}\right).$$

Next, eliminate K_2, K_3, K_4, K_5, and K_6 using Lemma 3. Bring all terms over the common denominator $(a_1+a_2)a_2(b_1+b_2)b_2fhK_1$. We get the following expression for the numerator.

$$\begin{aligned} N = {} & a_2^2b_2(b_1+b_2)c_2^2fh(e+j) \\ & - a_1a_2(b_1+b_2)c_2h\big(-b_2(c_2f(e-g)+c_1e(f+g)) + b_1c_1e(f+k)\big) \\ & - a_1^2b_2\big(b_2c_2^2fh(g+j) + b_1(c_1^2eg(e+h) + c_2^2fh(g+j) - c_1c_2eg(h+k))\big). \end{aligned}$$

It will suffice to prove that $N = 0$. In the expression for N, eliminate e, g, and k using Lemma 2. Factoring the resulting expression, we get

$$N = (a_2b_2c_2 - a_1b_1c_1) \times (\text{another factor}).$$

But $a_2b_2c_2 = a_1b_1c_1$ by Ceva's Theorem. Thus, $N = 0$. □

Theorem 2. *Let P be any point inside $\triangle ABC$. Then*

$$\frac{a_1b_1}{r_1}+\frac{a_2b_2}{r_3}+\frac{a_1b_2}{r_5}=\frac{a_1b_1}{r_2}+\frac{a_2b_2}{r_4}+\frac{a_1b_2}{r_6}.$$

Proof. This is just Theorem 1 applied to $\triangle BCA$. Alternatively, we can start with the equation in Theorem 1,

$$\frac{a_2c_2}{r_1}+\frac{a_2c_1}{r_3}+\frac{a_1c_1}{r_5}=\frac{a_2c_2}{r_2}+\frac{a_2c_1}{r_4}+\frac{a_1c_1}{r_6},$$

and apply Ceva's Theorem to get

$$\frac{a_1b_1c_1}{b_2r_1}+\frac{a_2c_1}{r_3}+\frac{a_1c_1}{r_5}=\frac{a_1b_1c_1}{b_2r_2}+\frac{a_2c_1}{r_4}+\frac{a_1c_1}{r_6}.$$

Multiplying both sides of the equation by b_2/c_1 gives the desired result. □

Similarly, we can apply Theorem 1 to $\triangle CAB$ to get the following.

Theorem 3. *Let P be any point inside $\triangle ABC$. Then*

$$\frac{b_1c_2}{r_1}+\frac{b_1c_1}{r_3}+\frac{b_2c_2}{r_5}=\frac{b_1c_2}{r_2}+\frac{b_1c_1}{r_4}+\frac{b_2c_2}{r_6}.$$

The result in Theorem 1 is so elegant that it is unlikely that it is true only because the complicated expression for N in the proof just happened to have $a_2b_2c_2-a_1b_1c_1$ as a factor.

Open Question 1. *Is there a simpler proof of Theorem 1 that gives more insight into why the result is true, without involving a lot of computation?*

The reader may wonder how I found Theorem 1. Here is the procedure that was used.

Procedure: I started with a triangle (Figure 2) whose six segments had symbolic lengths a_1, a_2, b_1, b_2, c_1, and c_2. Then, using Stewart's Theorem, I computed AD, BE, and CF in terms of these six variables. Then, using Van Aubel's Theorem for Triangles, I computed e, f, g, h, j, and k. I used these values to compute the s_i. Then, using Heron's Formula for the area of a triangle, I computed the K_i. Finally, using the formula $r=K/s$, I computed the r_i. The formulas were lengthy, involving many square roots, and the computations had to be done by computer. Then I guessed that there was a relationship involving r_i^t for some fixed t. I varied t from -6 to 6 (excluding $t=0$). For each t, I formed the six expressions $E_i=r_i^t$, for $i=1,2,...,6$. Now I picked values for a_1, a_2, b_1, c_1, and c_2 that were distinct primes. In particular, I chose $a_1=11$, $a_2=3$, $b_1=7$, $c_1=5$, and $c_2=13$. The value of b_2 was then determined by Ceva's Theorem. Next, I evaluated each of the E_i to 50 decimal places. I then used the Mathematica® function `FindIntegerNullVector` to see if there was any linear relationship with small integer coefficients between these six real numbers. For $t=-1$, the relationship

$$39E_1-39E_2+15E_3-15E_4+55E_5-55E_6=0$$

was found. Comparing the prime factorizations of the coefficients $39=3\cdot 13$, $15=3\cdot 5$, and $55=5\cdot 11$ against the chosen segments lengths ($a_1=11$, $a_2=3$,

$b_1 = 7$, $c_1 = 5$, and $c_2 = 13$) suggested that the coefficients were a_2c_2, a_2c_1, and a_1c_1. Surprisingly, b_1 and b_2 did not seem to be involved. Varying b_1 to have other values did not change the linear relationship found. The same pattern was observed when I tried other prime numbers for a_1, a_2, c_1, and c_2. This made the conjecture very plausible.

The same procedure was used to find the following theorem.

Theorem 4. *Let P be the incenter of $\triangle ABC$. Then*

$$\frac{b-c}{r_1}+\frac{b-c}{r_2}+\frac{c-a}{r_3}+\frac{c-a}{r_4}+\frac{a-b}{r_5}+\frac{a-b}{r_6}=0.$$

Proof. Let

$$S=(b-c)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+(c-a)\left(\frac{1}{r_3}+\frac{1}{r_4}\right)+(a-b)\left(\frac{1}{r_5}+\frac{1}{r_6}\right).$$

We want to prove that $S = 0$. Starting with the sides of $\triangle ABC$, (a, b, and c) we can compute a_1, a_2, b_1, b_2, c_1, and c_2 using the Angle Bisector Theorem (Euclid VI.3). For example, $a_1 = ac/(b + c)$. Then we use Stewart's Theorem to compute AD, BE, and CF. Then we again use the Angle Bisector Theorem to compute e, f, g, h, j, and k. We get

$$e=\frac{\sqrt{ac(a^2-b^2+2ac+c^2)}}{a+b+c} \quad\text{and}\quad f=\frac{b\sqrt{ac(a^2-b^2+2ac+c^2)}}{(b+c)(a+b+c)},$$

with similar expressions for g, h, j, and k. In the formula for S, replace r_i with K_i/s_i. Then, eliminate K_2, K_3, K_4, K_5, and K_6 using Lemma 3. We get

$$S=\frac{a^3(s_5-s_4)+b^3(s_1-s_6)+c^3(s_3-s_2)-bc^2s_1-a^2cs_3+ac^2s_4+ba^2s_6}{b(b+c)K_1}.$$

Now replace each s_i with the semiperimeter of triangle T_i using the values found for e, f, g, h, j, and k. After simplifying the resulting expression, we find that $S = 0$. □

The following theorem was suggested by Theorem 8.4 of [2].

Theorem 5. *Let P be the incenter of $\triangle ABC$. Then*

$$\frac{\cos\gamma}{r_1}+\frac{\cos\alpha}{r_3}+\frac{\cos\beta}{r_5}=\frac{\cos\beta}{r_2}+\frac{\cos\gamma}{r_4}+\frac{\cos\alpha}{r_6}$$

where $m\angle CAB = 2\alpha$, $m\angle ABC = 2\beta$, and $m\angle BCA = 2\gamma$ (Figure 4).

The proof is similar to the proof of Theorem 4, so is omitted. The cosines are computed using the Law Of Cosines.

Open Question 2. *Are there simpler proofs for Theorems 4 and 5 that don't involve a lot of computation?*

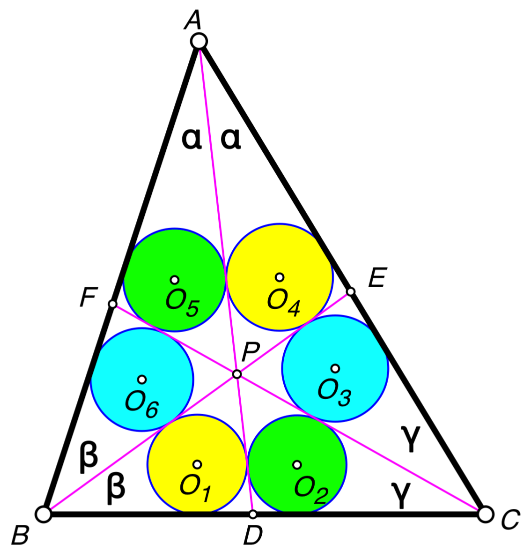

FIGURE 4.

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