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Coloring a Generalized Map

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Define a generalized map on the plane to be a partition of the plane into a finite family of pairwise disjoint, nonempty, connected sets. These sets are called countries. We say two countries are adjacent if their union is connected. Prove or disprove that such a map on the plane can be colored in five colors so that no two adjacent countries have the same color. What happens if we replace "connected" with "arcwise connected"?

Solution by Stanley Rabinowitz, Far Rockaway, N.Y. Finitely many colors will not suffice to color a generalized map. We show how to construct a generalized map with k countries each adjacent to each other for any k.

Let $A_1, A_2, \cdots, A_k$ be any k distinct points in the plane. Country E_i originally owns point A_i, $i=1, \cdots, k$. In step 1 of an expansion program, country E_1 extends its territory by claiming land along a continuous curve of finite length which starts at A_1 and does not intersect itself or A_j $(j\neq 1)$ and passes within a distance ϵ of each point A_j $(j\neq 1)$. In no way has this country enclosed any land, so all unoccupied points are accessible to any country by extensions along continuous curves. In step n $(n=1, 2, \cdots)$, country E_n (subscript reduced modulo k) extends its previously owned territory by claiming land along a continuous curve of finite length which starts at A_n or any other point it already owns and does not intersect itself or the property of any other country and which passes within $\epsilon/2^{n-1}$ of each point A_j $(j\not\equiv n$ modulo $k)$. This is possible because in no previous step has any unoccupied land been made inaccessible. If step n is completed in a time of $1/2^n$ minutes, after one minute each country will be disjoint, connected, and adjacent to each other country. We must merely verify that countries E_i and E_j are adjacent $(i\neq j)$. The expansion process assures that A_j is a limit point of country E_i. Hence $E_i\subset E_i\cup\{A_j\}\subset\overline{E}_i$, and so $E_i\cup\{A_j\}$ is connected. But $E_i\cup E_j$ is the union of the two intersecting connected sets E_j and $E_i\cup\{A_j\}$ and is therefore connected.

If we replace "connected" with "arcwise connected" in the definition, then five colors will suffice to color any generalized map. For pick a point in each country and connect those points in adjacent countries by a continuous simple curve lying wholly in their union, and not meeting any other such curve. These points will then be the vertices of a planar network and can thus be colored with five colors (so that endpoints of each arc are colored differently) by the dual of the usual five-color map theorem. Hence the generalized map can be colored with these five colors in an analogous way.