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Sum Formulae!

H-453 *Proposed by James E. Desmond, Pensacola Jr. College, Pensacola, FL (Vol. 29, no. 2, May 1991)*

Show that for positive integers m and n,

$$\frac{L_{(2m+1)n}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} L_{2nj} + (-1)^{m(n+1)}$$

and

$$\frac{F_{2mn}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)}.$$

(See solution below.)

Solution by Stanley Rabinowitz, Westford, MA

Lemma:

$$S(n,\ a,\ b,\ r)\ \equiv \sum_{j=1}^{n} r^j F_{aj+b} = \frac{(-1)^a r^{n+2} F_{an+b} - r^{n+1} F_{a(n+1)b} - (-1)^a r^2 F_b + rF_{a+b}}{(-1)^a r^2 - rL_a + 1}.$$

Proof: Let

$$G(x,\ n)\ \equiv \sum_{j=1}^{n} x^j = x\left(\frac{x^n - 1}{x - 1}\right).$$

Now

$$r^j F_{aj+b} = r^j\left(\frac{\alpha^{aj+b} - \beta^{aj+b}}{\sqrt{5}}\right) = \frac{\alpha^b}{\sqrt{5}}(r\alpha^a)^j - \frac{\beta^b}{\sqrt{5}}(r\beta^a)^j .$$

Thus,

$$S(n,\ a,\ b,\ r) = \frac{\alpha^b}{\sqrt{5}}\,G\,(r\alpha^a,\ n) - \frac{\beta^b}{\sqrt{5}}\,G\,(r\beta^a,\ n)$$

$$= \frac{\alpha^b}{\sqrt{5}}\ r\alpha^a\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) - \frac{\beta^b}{\sqrt{5}}\ r\beta^a\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)$$

$$= \frac{r}{\sqrt{5}}\left[\alpha^{a+b}\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) - \beta^{a+b}\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)\right]$$

$$= \frac{r}{\sqrt{5}}\left[\frac{\alpha^{a+b}(r\beta^a - 1)\,(r^n\alpha^{an} - 1) - \beta^{a+b}(r\alpha^a - 1)\,(r^n\beta^{an} - 1)}{(r\alpha^a - 1)\,(r\beta^a - 1)}\right]$$

$$= \frac{r}{\sqrt{5}}\left[\frac{r^{n+1}\,(\beta^a\alpha^{a(n+1)+b} - \alpha^a\beta^{a(n+1)+b}) - r^n(\alpha^{a(n+1)+b} - \beta^{a(n+1)+b}) - r(\alpha^{a+b}\beta^a - \alpha^a\beta^{a+b}) + \alpha^{a+b} - \beta^{a+b}}{r^2(\alpha\beta)^a - r(\alpha^a + \beta^a) + 1}\right]$$

$$= \frac{r}{\sqrt{5}}\left[\frac{r^{n+1}(\alpha\beta)^a\,(\alpha^{an+b} - \beta^{an+b}) - r^n(\alpha^{a(n+1)+b} - \beta^{a(n+1)+b}) - r(\alpha\beta)^a\,(\alpha^b - \beta^b) + (\alpha^{a+b} - \beta^{a+b})}{(\alpha\beta)^a r^2 - r(\alpha^a + \beta^a) + 1}\right]$$

$$= r\left[\frac{r^{n+1}(-1)^a F_{an+b} - r^n F_{a(n+1)+b} - r(-1)^a F_b + F_{a+b}}{(-1)^a r^2 - rL_a + 1}\right]$$

$$= \frac{(-1)^a r^{n+2} F_{an+b} - r^{n+1} F_{a(n+1)+b} - (-1)^a r^2 F_b + rF_{a+b}}{(-1)^a r^2 - rL_a + 1}$$

which was to be proved.

Using this lemma, we have

$$\sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)}$$

$$= (-1)^{(n+1)m} S(m,\ 2n,\ -n,\ (-1)^{n+1})$$

$$= (-1)^{(n+1)m}\ \frac{(-1)^{(n+1)(m+2)}\ F_{2mn-n} - (-1)^{(n+1)(m+1)}\ F_{2n(m+1)-n} - F_{-n} + (-1)^{n+1}F_n}{2 - (-1)^{n+1}L_{2n}}$$

$$= \frac{F_{n(2m-1)} + (-1)^n F_{n(2m+1)}}{2 + (-1)^n L_{2n}}$$

where we have used the fact that $F_{-n} = (-1)^{n+1}F_n$.

Thus, it remains to prove that our answer,

$$(1)\qquad \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)} = \frac{F_{n(2m-1)} + (-1)^n F_{n(2m+1)}}{2 + (-1)^n L_{2n}}$$

is equivalent to the proposer's answer of F_{2mn}/L_n. Cross multiplying, we see that this would be equivalent to showing that

$$(2) \qquad F_{n(2m-1)}L_n + (-1)^n F_{n(2m+1)} = 2F_{2mn} + (-1)^n F_{2mn} L_{2n}.$$

Applying the well-known identity,

$$F_x L_y = F_{x+y} + (-1)^y F_{x-y}$$

to equation (2), we find that all the terms drop out; hence, equation (2) is true. Thus, our answer (1) is equivalent to the proposer's answer.

In the same manner, we can prove a similar lemma for the Lucas numbers:

$$T(n, a, b, r) \equiv \sum_{j=1}^{n} r^j L_{aj+b}$$

$$= \alpha^b G(r\alpha^a, n) + \beta^b G(r\beta^a, n)$$

$$= \alpha^b r\alpha^a\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) + \beta^b r\beta^a\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)$$

$$= r\left[\alpha^{a+b}\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) + \beta^{a+b}\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)\right]$$

$$= r\left[\frac{\alpha^{a+b}(r\beta^a - 1)(r^n\alpha^{an} - 1) + \beta^{a+b}(r\alpha^a - 1)(r^n\beta^{an} - 1)}{(r\alpha^a - 1)(r\beta^a - 1)}\right]$$

$$= r\left[\frac{\begin{array}{c} r^{n+1}(\beta^a\alpha^{a(n+1)+b} + \alpha^a\beta^{a(n+1)+b}) - r^n(\alpha^{a(n+1)+b} + \beta^{a(n+1)+b}) \\ - r(\alpha^{a+b}\beta^a + \alpha^a\beta^{a+b}) + (\alpha^{a+b} + \beta^{a+b}) \end{array}}{r^2(\alpha\beta)^a - r(\alpha^a + \beta^a) + 1}\right]$$

$$= r\left[\frac{\begin{array}{c} r^{n+1}(\alpha\beta)^a(\alpha^{an+b} + \beta^{an+b}) - r^n(\alpha^{a(n+1)+b} + \beta^{a(n+1)+b}) \\ - r(\alpha\beta)^a(\alpha^b + \beta^b) + (\alpha^{a+b} + \beta^{a+b}) \end{array}}{(\alpha\beta)^a r^2 - r(\alpha^a + \beta^a) + 1}\right]$$

$$= \frac{(-1)^a r^{n+2} L_{an+b} - r^{n+1} L_{a(n+1)+b} - (-1)^a r^2 L_b + rL_{a+b}}{(-1)^a r^2 - rL_a + 1}.$$

Using this result, we have

$$\sum_{j=1}^{m} (-1)^{(n+1)(m-j)} L_{2nj}$$

$$= (-1)^{(n+1)m} T(m, 2n, 0, (-1)^{n+1})$$

$$= (-1)^{(n+1)m}\left[\frac{(-1)^{(n+1)(m+2)} L_{2mn} - (-1)^{(n+1)(m+1)} L_{2n(m+1)} - L_0 + (-1)^{n+1} L_{2n}}{2 - (-1)^{n+1} L_{2n}}\right]$$

$$= \frac{L_{2mn} + (-1)^n L_{2n(m+1)} - 2(-1)^{(n+1)m} + (-1)^{(n+1)(m+1)} L_{2n}}{2 + (-1)^n L_{2n}}.$$

To show that our answer is equivalent to the proposer's, we must show that

$$\frac{L_{(2m+1)n}}{L_n} - (-1)^{m(n+1)} = \frac{L_{2mn} + (-1)^n L_{2n(m+1)} - 2(-1)^{(n+1)m} + (-1)^{(n+1)(m+1)} L_{2n}}{2 + (-1)^n L_{2n}}$$

or, equivalently,

$$2L_{n(2m+1)} - 2(-1)^{m(n+1)}L_n + (-1)^n L_{2n}L_{n(2m+1)} - (-1)^{m(n+1)+n}L_nL_{2n}$$
$$= L_nL_{2mn} + (-1)^n L_nL_{2n(m+1)} - 2(-1)^{m(n+1)}L_n + (-1)^{(n+1)(m+1)} L_nL_{2n}.$$

Again, this falls out by applying the well-known identity,

$$L_xL_y = L_{x+y} + (-1)^y L_{x-y}.$$