Reprinted from the Pi Mu Epsilon Journal 7.8(1983)548–549.

514. [Spring 1982] *Proposed by Raymond E. Spaulding, Radford University, Radford, Virginia.*

Let $A_1A_2A_3 \ldots A_n$ be a regular polygon where $A_{n+j} = A_j$ and $A_iA_{i+1} = 1$. Let B_i be a point on the segment A_iA_{i+1} where $A_iB_i = x$. Let C_i be the point where A_iB_{i+1} intersects $A_{i+1}B_{i+2}$. Find the area of a regular polygon $C_1C_2C_3 \ldots C_n$ in terms of n and x.

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

Let $A_2B_3 = k$, $A_3C_2 = A_2C_1 = y$, and $C_2B_3 = z$. By symmetry, $\angle B_3A_2A_3 = \angle C_2A_3B_3$, so triangles $B_3A_3A_2$ and $B_3C_2A_3$ are similar (by angle-angle). Thus

$$\frac{x}{1} = \frac{z}{y} \text{ and } \frac{k}{1} = \frac{x}{y}, \qquad \text{so} \qquad y = \frac{x}{k} \text{ and } z = \frac{x^2}{k}$$

and

$$C_1C_2 = A_2B_3 - A_2C_1 - C_2B_3 = k - \frac{x}{k} - \frac{x^2}{k} = \frac{k^2 - x - x^2}{k}.$$

Applying the law of cosines to triangle $B_3A_2A_3$, we find that

$$k^2 = 1 + x^2 + 2x\cos\theta$$

where $\theta = 2\pi/n$ is the exterior angle of the regular n-gon. The area of a regular n-gon of side s is $ns^2/(4\tan(\theta/2))$, so the area of the regular n-gon with side C_1C_2 is, where $\theta = 2\pi/n$,

$$\frac{n(k^2 - x - x^2)^2}{4k^2\tan(\theta/2)} = \frac{n(1 + 2x\cos\theta - x)^2}{4(1 + 2x\cos\theta + x^2)\tan(\theta/2)}.$$