A Useful Trigonometric Substitution

Stanley Rabinowitz
Digital Equipment Corporation
Nashua, NH

In this article, we will show that the substitution

$$t = \tan\frac{x}{2}$$

is very useful in solving certain types of trigonometry problems.

First, let us investigate formulas for the tangent of a half angle. One formula that is frequently taught is

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}.$$

This formula is frequently useful when you are solving for $x/2$ and know which quadrant the angle x falls in. Unfortunately, the formula is not very useful in proving trigonometric identities or inequalities because of the ambiguity of the $\pm$ sign. In fact, depending on the angle x, sometimes the positive square root should be taken, and sometimes the negative square root.

For theoretical (as opposed to computational) problems, an exact formula is required, one in which there is no ambiguity in sign. For this purpose, a good formula to know is

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}.$$

This formula can be derived from the familiar formulas for the sine and cosine of a sum and difference:

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \tag{1}$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y \tag{2}$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \tag{3}$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y. \tag{4}$$

Adding equations (1) and (2) and also (3) and (4) gives

$$\sin(x+y) + \sin(x-y) = 2\sin x\cos y$$

$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$$

or letting $x+y=2A$ and $x-y=2B$, we find

$$\sin 2A + \sin 2B = 2\sin(A+B)\cos(A-B)$$

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$$\cos 2A + \cos 2B = 2\cos(A+B)\cos(A-B).$$

Letting $x = 2A$, $y = 2B$ and dividing these two equations gives

$$\tan\frac{x+y}{2} = \frac{\sin x + \sin y}{\cos x + \cos y}.$$

This, by itself, is a very useful formula to know. Furthermore, letting $y = 0$ gives our half-angle formula

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}. \tag{5}$$

Also, the familiar identity $\sin^2 x + \cos^2 y = 1$ may be written int he form

$$\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

which shows that an equivalent half-angle formula is

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}. \tag{6}$$

Letting $t = \tan\frac{x}{2}$, we can solve equations (5) and (6) for $\sin x$ and $\cos x$ in terms of t. We get

$$\sin x = \frac{2t}{1+t^2} \tag{7}$$

$$\cos x = \frac{1-t^2}{1+t^2}. \tag{8}$$

Now we see why the substitution $t = \tan\frac{x}{2}$ is so useful. It lets us express both $\sin x$ and $\cos x$ as rational functions of the single variable, t.

Let us now see how this substitution can be useful in solving various problems.

Example 1 (Equations). J. T. Groenman asked in [1] to find all x in $(0, 2\pi)$ satisfying

$$176\cos x + 64\sin x = 75\cos 2x + 80\sin 2x + 101.$$

We can solve any problem like this with the aid of the substitution $t = \tan\frac{x}{2}$. If the equation we are trying to solve contains any trigonometric terms other than sin and cos (such as tan, cot, sec, or csc), we first express these in terms of sin and cos. Then, if there are any expressions involving multiple angles (or sums or differences), we expand these out using the appropriate formulas. In this example, we use the formulas for the sin and cos of $2x$ to get

$$75(\cos^2 x - \sin^2 x) + 160\sin x\cos x - 64\sin x - 176\cos x + 101 = 0.$$

Now we apply our substitution, using relations (7) and (8) to transform $\sin x$ and $\cos x$ into expressions in terms of t. We get

$$\frac{352t^4 - 448t^3 - 248t^2 + 192t}{(1+t^2)^2} = 0.$$

Since the numerator must be equal to 0, and dividing out by $8t$, we find that

$$44t^3 - 56t^2 - 31t + 24 = 0.$$

Any rational solution to this must have a denominator that divides 44. Trying a denominator of 2, we find that $t = 1/2$ is a solution. Dividing out the factor $(2t-1)$ gives us a quadratic that we can easily factor, and so we find that the complete factorization is

$$t(2t-1)(2t-3)(11t+8) = 0.$$

Thus, the complete solution is

$$x = \begin{cases} 0 \\ 2\arctan(1/2) \\ 2\arctan(3/2) \\ 2\arctan(-8/11) \end{cases}$$

where $\arctan z$ denotes the angle in $[0, \pi)$ whose tangent is z.

Example 2 (Inequalities). Find the range of x in the interval $[0, 2\pi)$ for which

$$8\sin x + \cos x > 4.$$

Applying the substitutions in (7) and (8) show that this problem is equivalent to

$$\frac{(t-3)(5t-1)}{1+t^2} < 0.$$

The quantity on the left is 0 when $t = 3$ or when $t = 1/5$. Since $1 + t^2$ is always positive, the sign of the left-hand side depends only on the sign of $(t-3)(t-1/5)$. Between $1/5$ and 3 this expression is negative; elsewhere it is positive. Thus t lies in the interval $(1/5, 3)$ and hence x lies in the interval $(2\arctan\frac{1}{5}, 2\arctan 3)$.

Example 3 (Integration). For those of you who know some calculus, the substitution $t = \tan\frac{x}{2}$ will let you integrate any rational function of $\sin x$ and $\cos x$ by changing it to a rational function of t. This latter expression can then be integrated by expanding into partial fractions. For details, consult a calculus textbook such as [3].

As we have seen, the subsitution $t = \tan\frac{x}{2}$ is a very effective method of solving certain problems. However, it should only be used as a last resort because its use can involve very complicated expressions. Its effectiveness makes it ideal for computer applications. For hand calculation, it is advisable to first look for other tricks or substitutions. Ingenuity will normally lead to a simpler solution than brute force. The following example illustrates this point.

Example 4. Murray Klamkin asked in [2] to find all x in $(0, 2\pi)$ satisfying

$$81\sin^{10} x_+ \cos^{10} x = \frac{81}{256}.$$

In this case, the simpler substitution $t = \sin^2 x$, $1-t = \cos^2 x$ can be used. This substitution may be used any time that all the exponents of $\sin x$ and $\cos x$ are even. In this case, we get

$$81t^5 + (1-t)^5 = \frac{81}{256}$$

which expands out to

$$4096t^5 + 256t^4 - 512t^3 + 512t^2 - 256t + 35 = 0.$$

The powers of 2 appearing as coefficients suggests that we make the further substitution $t = T/4$ which yields

$$4T^5 + T^4 - 8T^3 + 32T^2 - 64T + 35 = 0.$$

In this form, $T = 1$ is seen to be a solution (twice) and the polynomial factors as

$$(T-1)^2(4T^3 + 9T^2 + 6T + 35) = 0.$$

Since $T = 4\sin^2 x$ is always non-negative, $4T^3 + 9T^2 + 6T + 35$ can never be 0 and so $T = 1$ is the only solution. This yields $t = \frac{1}{4}$ and $\sin x = \pm\frac{1}{2}$. Thus, $x = \pi/6$, $5\pi/6$, $7\pi/6$, or $11\pi/6$.

REFERENCES

[1] J. T. Groenman, "Problem 1020", *Crux Mathematicorum.* **11**(1985)51.
[2] Murray S. Klamkin, "Problem D–3", *AMATYC Review.* **4**(1983)63.
[3] George B. Thomas, Jr., *Calculus and Analytic Geometry* (3rd edition). Addison-Wesley Publishing Company, Inc. Reading: 1960.