Algorithmic Manipulation of Second Order Linear Recurrences

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1. Introduction.

Consider the second order linear recurrences defined by

$$u_{n+2} = Pu_{n+1} - Qu_n, \qquad u_0 = 0, \quad u_1 = 1, \tag{1}$$

$$v_{n+2} = Pv_{n+1} - Qv_n, \qquad v_0 = 2, \quad v_1 = P, \tag{2}$$

and

$$w_{n+2} = Pw_{n+1} - Qw_n, \qquad w_0,\ w_1 \text{ arbitrary.} \tag{3}$$

The sequences $\langle u_n \rangle$ and $\langle v_n \rangle$ were studied extensively by Lucas [17], and the sequence $\langle w_n \rangle$ was popularized by Horadam [10], [11], [12], and also studied by Zeitlin [23], [26], [27]. The sequence $\langle u_n \rangle$ is known as the fundamental Lucas sequence and the sequence $\langle v_n \rangle$ is known as the primordial Lucas sequence.

The relationship between w_n and the pair of sequences u_n and v_n is well known. Horadam [10] gives several formulas for w_n:

$$w_n = \frac{(2w_1 - Pw_0)u_n + w_0 v_n}{2} \tag{4}$$

$$w_n = (w_1 - Pw_0)u_n + w_0 u_{n+1} \tag{5}$$

$$w_n = w_1 u_n - Qw_0 u_{n-1}. \tag{6}$$

In [19], it was shown that Algorithm `LucasSimplify` could be used to prove any polynomial identity involving expressions of the form u_{an+b} and v_{an+b}. Since w_n can be expressed in terms of u_n and v_n, this means that we can algorithmically prove any polynomial identity involving expressions of the form w_{an+b} using Algorithm `LucasSimplify`.

However, Algorithm `LucasSimplify`, when applied to an expression involving w's will return a simplified expression involving u's and v's. Since it may be of interest to get results in terms of w's, we will now develop new algorithms that can be used to transform expressions involving w's from one form to another.

For example, Melham and Shannon [18] found an "addition formula" for simplifying w_{m+n}:

$$w_{m+n} = \frac{(2w_{m+1} - Pw_m)u_n + w_m v_n}{2}.$$

Unfortunately, this formula involves the sequences $\langle u_n \rangle$ and $\langle v_n \rangle$. We call an identity *impure* if it contains terms involving u's or v's. Otherwise, if the identity only involves w's, we call it *pure*. It is our goal to find a pure formula for w_{n+m} and related expressions.

Reprinted from The Fibonacci Quarterly, 37(1999)162–177

2. Overview.

Two expressions occur frequently enough, that we shall give them names:

$$D = P^2 - 4Q \qquad \text{and} \qquad e = w_0 w_2 - w_1^2. \tag{7}$$

Since $w_2 = Pw_1 - Qw_0$, an equivalent formula for e is

$$e = Pw_0 w_1 - Qw_0^2 - w_1^2. \tag{8}$$

The quantity D is the discriminant of the characteristic equation for the recurrence, and the quantity e is known as the characteristic number of the sequence [2], [1]. Throughout this paper, we shall assume that

$$Q \neq 0, \qquad D \neq 0, \qquad \text{and} \qquad e \neq 0. \tag{9}$$

In section 3 we develop the Purification Theorem, which shows how to transform impure identities into pure identities. In subsequent sections, we then find the pure analogs (for w_n) of all the classic identities known for u_n and v_n, either by giving a reference to the literature where the pure identity was discovered, or by deriving the pure identity ourselves. If a simpler proof of the result can be given without using the Purification Theorem, then we present the simpler proof. We then give algorithms that allow pure expressions to be transformed from one form to another.

3. The Purification Theorem.

To achieve our goal of finding pure identities, we need only express u_n and v_n in terms of members of the sequence $\langle w_n \rangle$.

Theorem 1 (The Purification Theorem). Any identity involving u's, v's, and w's can be transformed into a pure identity (involving only w's). In particular,

$$\begin{aligned} u_n &= \frac{w_0 w_{n+1} - w_1 w_n}{e}, \\ v_n &= \frac{(Pw_0 - 2w_1)w_{n+1} - (2Qw_0 - Pw_1)w_n}{e}. \end{aligned} \tag{10}$$

Proof: Algorithm `LucasSimplify` allows us to express both w_n and w_{n+1} in terms of u_n and v_n. Solving these two equations for u_n and v_n gives us formula (10). Thus any expression involving u's and v's can be transformed into expressions involving w's. $\square$

4. The Addition Formula.

The addition formulas for u_n and v_n are well known:

$$\begin{aligned} u_{m+n} &= \frac{u_m v_n + u_n v_m}{2} \\ v_{m+n} &= \frac{v_m v_n + D u_m u_n}{2}. \end{aligned} \tag{11}$$

We would like to find a similar formula for w_{m+n}. Horadam [10] gives several such formulas:

$$\begin{aligned} w_{n+m} &= u_m w_{n+1} - Q u_{m-1} w_n \qquad (12) \\ w_{n+m} &= (u_{m+1} - P u_m) w_n + u_m w_{n+1} \\ w_{n+m} &= w_m u_{n+1} - Q w_{m-1} u_n \\ w_{n+m} &= w_n u_{m+1} - Q w_{n-1} u_m \\ w_{n+m} &= w_{m-j} u_{n+j+1} - Q w_{m-j-1} u_{n+j} \\ w_{n+m} &= w_{n+j} u_{m-j+1} - Q w_{n+j-1} u_{m-j} \end{aligned}$$

however, these are all impure.

Applying `LucasSimplify` to u_{m-1} gives $u_{m-1} = (P u_m - v_m)/2Q$. Substituting this value of u_{m-1} into Horadam's addition formula (12) and then applying the Purification Theorem gives us:

$$w_{n+m} = \frac{(w_0 w_{m+1} - w_1 w_m) w_{n+1} - (w_1 w_{m+1} - w_2 w_m) w_n}{e}.$$

We state this in another form in the following theorem.

Theorem 2 (The Addition Formula for w). For all integers n and m,

$$w_{n+m} = -\frac{1}{e} \begin{vmatrix} w_0 & w_1 & w_m \\ w_1 & w_2 & w_{m+1} \\ w_n & w_{n+1} & 0 \end{vmatrix}. \tag{13}$$

5. The Negation Formula.

Having found the addition formula entirely in terms of w's, we now proceed to express all the other standard formulas in the same manner.

Horadam [10] expressed the negation formula in the following ways:

$$w_{-n} = Q^{-n}(w_0 u_{n+1} - w_1 u_n)$$

$$w_{-n} = Q^{-n}(w_0 v_n - w_n).$$

He also found the interesting formula $\quad w_n w_{-n} = w_0^2 + e Q^{-n} u_n^2$.

Unfortunately, these formulas are all impure. We can use the Purification Theorem to remove the u's and v's to arrive at a pure negation formula.

Theorem 3 (The Negation Formula for w). For all integers n,

$$w_{-n} = \frac{(w_1^2 - Qw_0^2)w_n + w_0(Pw_0 - 2w_1)w_{n+1}}{eQ^n}$$
$$= -\frac{1}{eQ^n}\begin{vmatrix} w_{-1} & w_0 & w_1 \\ w_0 & w_1 & Qw_0 \\ w_n & w_{n+1} & 0 \end{vmatrix}. \tag{14}$$

Solving equation (14) for w_{n+1} gives us a useful formula that allows one to express w_{n+1} in terms of w_n and w_{-n}.

Theorem 4 (The Symmetrization Formula). For all integers n,

$$w_{n+1} = \frac{(w_1^2 - Qw_0^2)w_n - eQ^n w_{-n}}{w_0(2w_1 - Pw_0)} \tag{15}$$

provided that the denominator is not 0.

6. The Subtraction Formula.

Melham and Shannon [18] expressed the subtraction formula in the following form:

$$w_{m-n} = \frac{w_m u_{n+1} - w_{m+1} u_n}{Q^n}.$$

Again, this is an impure formula. We can now combine the negation formula with the addition formula to get a pure subtraction formula.

Theorem 5 (The Subtraction Formula for w). For all integers n and m,

$$w_{n-m} = -\frac{1}{eQ^m}\begin{vmatrix} w_{-1} & w_0 & w_{n+1} \\ w_0 & w_1 & Qw_n \\ w_m & w_{m+1} & 0 \end{vmatrix}. \tag{16}$$

Proof: Horadam [10] found $w_{n+m} + Q^m w_{n-m} = w_n v_m$. Solve for w_{n-m} and then expand w_{n+m} by the addition formula and express v_m in terms of w_m and w_{m+1} by the Purification Theorem. Upon simplifying, we get the desired result. □

7. The Binet Form.

The Binet form (see [10]) for w_n is given by the following theorem.

Theorem 6 (The Binet Form). If r_1 and r_2 are the roots of the characteristic equation

$$x^2 - Px + Q = 0,$$

then

$$w_n = Ar_1^n + Br_2^n \tag{17}$$

where

$$A = \frac{w_1 - w_0 r_2}{r_1 - r_2} \qquad \text{and} \qquad B = \frac{w_0 r_1 - w_1}{r_1 - r_2}. \tag{18}$$

This generalizes the result for Fibonacci numbers found by Binet [3].

Note that $r_1 \neq r_2$ since $P^2 - 4Q \neq 0$. One should also note that

$$AB = \frac{e}{D} \qquad \text{and} \qquad A + B = w_0. \tag{19}$$

We also have

$$r_1 - r_2 = \sqrt{D}. \tag{20}$$

Since $w_{n+1} = (Ar_1)r_1^n + (Br_2)r_2^n$, we can solve the system consisting of this equation and equation (17) for r_1^n and r_2^n. We get the following:

$$r_1^n = \frac{w_{n+1} - r_2 w_n}{w_1 - r_2 w_0} \qquad \text{and} \qquad r_2^n = \frac{w_{n+1} - r_1 w_n}{w_1 - r_1 w_0}. \tag{21}$$

These formulas may be used to replace powers of r_1 and r_2 (with variable exponents) by simpler expressions involving r_1 and r_2.

If we let $x_n = w_{n+1} - Qw_{n-1}$, then x_n may be considered to be a companion sequence to w_n, in the same way that v_n is the companion of u_n. A little computation shows that

$$r_1^n = \frac{1}{2}(x_n + w_n(r_1 - r_2))/(w_1 - r_2 w_0).$$

This gives us the following theorem, since $(r_1^n)^k = r_1^{kn}$.

Theorem 7 (Demoivre's Formula for w_n). If $x_n = w_{n+1} - Qw_{n-1}$ and $c = w_1 - r_2 w_0 \neq 0$, then for all integers $k > 0$,

$$\left(\frac{x_n + w_n\sqrt{D}}{2c}\right)^k = \frac{x_{kn} + w_{kn}\sqrt{D}}{2c}. \tag{22}$$

This theorem is so named because of its resemblance to Demoivre's trigonometry formula. If $\langle w_n \rangle = \langle u_n \rangle$, we have

$$\left(\frac{v_n + u_n\sqrt{D}}{2}\right)^k = \frac{v_{kn} + u_{kn}\sqrt{D}}{2}.$$

8. Simson's Formula.

In 1753, Robert Simson [21] found the formula

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

The analog for the sequence $\langle w_n \rangle$ was found by Horadam [10]:

Theorem 8 (Simson's Formula for w). For all integers n,

$$w_{n-1}w_{n+1} - w_n^2 = Q^{n-1}e. \tag{23}$$

Theorem 8 can also be expressed in the following manner:

$$w_n w_{n+2} - w_{n+1}^2 = Q^n e. \tag{24}$$

Horadam [10] also found the following generalization of Simson's Formula:

Theorem 9 (Catalan's Identity for w). For all integers n and r,

$$w_{n+r}w_{n-r} - w_n^2 = eQ^{n-r}u_r^2. \tag{25}$$

This generalizes a result found by Catalan for Fibonacci Numbers in 1886 [4].

The determinant form of Simson's Theorem is

$$\begin{vmatrix} w_{n-1} & w_n \\ w_n & w_{n+1} \end{vmatrix} = Q^n e. \tag{26}$$

Horadam [10] generalized this to

$$\begin{vmatrix} w_{n-r} & w_{n+t} \\ w_n & w_{n+r+t} \end{vmatrix} = Q^{n-r}eu_r u_{r+t} \tag{27}$$

which extends a result for generalized Fibonacci numbers found by Tagiuri [22] in 1901. Horadam and Shannon [13] expressed this as

$$\begin{vmatrix} w_{n+r+s} & w_{n+s} \\ w_{n+r} & w_n \end{vmatrix} = eQ^n u_r u_s. \qquad (28).$$

In this form, it generalizes a 1960 result for Fibonacci numbers [7]. The special case of identity (27) when $r = 1$, $n = a + 1$, and $t = b - a - 1$ is of interest:

Theorem 10 (D'Ocagne's Identity for w). For all integers a and b,

$$\begin{vmatrix} w_a & w_b \\ w_{a+1} & w_{b+1} \end{vmatrix} = Q^a e u_{b-a}. \tag{29}$$

This generalizes a result found by d'Ocagne for Fibonacci Numbers in 1885 [6]. The special case of formula (27) when $n = a + r$ and $t = b - a - r$ is also of interest:

$$\begin{vmatrix} w_a & w_b \\ w_{a+r} & w_{b+r} \end{vmatrix} = Q^a e u_r u_{b-a}. \tag{30}$$

This formulation (with $a = n$ and $b = n + s$) comes from [13]. Catalan's identity can be expressed as $w_{n+r}w_{n-r} = w_n^2 + eQ^{n-r}u_r^2$.

Letting $r = 1$ and $r = 2$ in this formula and multiplying the results together yields a polynomial with w_n^4 and w_n^2 terms. The w_n^2 term can be made to vanish in the case when $Q = -P^2$. This gives the following result.

Theorem 11. If $P^2 + Q = 0$, then

$$w_n^4 - w_{n-2}w_{n-1}w_{n+1}w_{n+2} = (eQ^{n-1})^2. \tag{31}$$

This generalizes the identity $F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1$ that was stated by Gelin in 1880 and proved by Cesàro [5]. For another generalization of the Gelin-Cesàro Identity, see [13].

Letting $r = n$ in formula (25) gives another interesting case.

Theorem 12. For all integers n,

$$w_0 w_{2n} - w_n^2 = e u_n^2. \tag{32}$$

Gilbert [8] found an interesting pure formula in the form of a 3×3 determinant:

Theorem 13. For all integers a, b, c, x, y, and z,

$$\begin{vmatrix} w_{a+x} & w_{a+y} & w_{a+z} \\ w_{b+x} & w_{b+y} & w_{b+z} \\ w_{c+x} & w_{c+y} & w_{c+z} \end{vmatrix} = 0. \tag{33}$$

9. Change of Basis.

We often wish to change an expression involving w_n and w_{n+1} into one involving w_{n+a} and w_{n+b} for two distinct integers a and b.

Theorem 14. For all integers n,

$$\begin{pmatrix} w_n \\ w_{n+1} \end{pmatrix} = \frac{1}{w_{a+1}w_b - w_a w_{b+1}} \begin{pmatrix} w_1 & -w_0 \\ w_2 & -w_1 \end{pmatrix} \begin{pmatrix} w_b & -w_a \\ w_{b+1} & -w_{a+1} \end{pmatrix} \begin{pmatrix} w_{n+a} \\ w_{n+b} \end{pmatrix}. \tag{34}$$

Proof: Use the Addition Formula to express w_{a+1} and w_{b+1} in terms of w_n and w_{n+1}. This gives two equations in the two variables w_n and w_{n+1}. We can thus solve for these variables. Putting the result in matrix form gives us the above formula. $\square$

Note that the basis change is not always possible. The denominator can be written in the form $-Q^a e u_{b-a}$ by formula (29). Thus, the change of basis is possible if and only if $u_{b-a} \neq 0$.

10. The Fundamental Identity.

Theorem 15 (The Fundamental Identity). The fundamental identity connecting w_n and w_{n+1} is

$$P w_n w_{n+1} - Q w_n^2 - w_{n+1}^2 = eQ^n. \tag{35}$$

Proof: This follows immediately from Formula (24), after replacing w_{n+2} by the value given in equation (3). $\square$

This result is not new; it is equivalent to Simson's Theorem. If a is a constant, then the fundamental identity connecting w_n and w_{n+a} is

$$v_a w_n w_{n+a} - Q^a w_n^2 - w_{n+a}^2 = eQ^n u_a^2. \tag{36}$$

This was obtained by using formula (34) on the fundamental identity, changing the basis from $\{w_n, w_{n+1}\}$ to $\{w_n, w_{n+a}\}$. Changing n to x and $n+a$ to y gives the fundamental identity connecting w_x and w_y:

$$v_{y-x} w_x w_y - Q^{y-x} w_x^2 - w_y^2 = eQ^x u_{y-x}^2. \tag{37}$$

11. Removal of P and Q.

It is occasionally useful to be able to remove the quantity P from an expression. If the expression is a polynomial in the variables P and w_{c_i} where the c_i are constants and if P always occurs in a product with one of the w_{c_i}, then we can use the following results to accomplish our goal.

Theorem 16. If k is a positive integer, then

$$P^k w_0 = \sum_{j=0}^{k} \binom{k}{j} Q^j w_{k-2j}. \tag{38}$$

Applying the Translation Theorem (see Section 17) yields:

Theorem 17. If k is a positive integer and r is an integer, then

$$P^k w_r = \sum_{j=0}^{k} \binom{k}{j} Q^j w_{k+r-2j}. \tag{39}$$

We also have:

Theorem 18. If k is a positive integer and r is an integer, then

$$Q^k w_r = (-1)^k \sum_{j=0}^{k} \binom{k}{j} (-P)^j w_{2k+r-j} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P^{k-j} w_{k+r+j}. \tag{40}$$

12. The Double Argument Formula.

Horadam [10] found the double argument formula in the following form

$$w_{2n} = (-Q)^n \sum_{j=0}^{n} \binom{n}{j} (-\frac{P}{Q})^{n-j} w_{n-j}. \tag{41}$$

However, this is not a closed form.

Horadam also found a closed form (for $w_0 \neq 0$): $w_{2n} = (w_n^2 + eu_n^2)/w_0$. Shannon and Horadam [20] found the double argument formula in the following form: $w_{2n} = v_n w_n - w_0 Q^n$.

Unfortunately, both these formulas are impure. To get a pure formula, let $m = n$ in the addition formula. We obtain the following result.

Theorem 19 (The Double Argument Formula for w). For all integers n,

$$w_{2n} = \frac{w_2 w_n^2 - 2w_1 w_n w_{n+1} + w_0 w_{n+1}^2}{e} = \frac{-\begin{vmatrix} w_0 & w_1 & w_n \\ w_1 & w_2 & w_{n+1} \\ w_n & w_{n+1} & 0 \end{vmatrix}}{\begin{vmatrix} w_0 & w_1 \\ w_1 & w_2 \end{vmatrix}}. \tag{42}$$

13. Formulas for w_{kn}.

To find expressions for w_{kn} where k is a positive integer constant, you can use the recurrence found by Zeitlin [24]:

$$w_{kn} = v_n w_{(k-1)n} - Q^n w_{(k-2)n}, \qquad k \geq 2. \tag{43}$$

Lee [16] found a more direct formula for multiple argument reduction. For $k > 1$,

$$w_{kn} = w_n S(k) - w_0 Q^n S(k-1) \tag{44}$$

where

$$S(k) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-j-1}{j} (-Q^n)^j v_n^{k-2j-1}. \tag{45}$$

Jarden [14] found the following interesting formula:

$$w_{kn+s} = \sum_{i=0}^{k} \binom{k}{i} u_n^i (-Qu_{n-1})^{k-i} w_{i+s}. \tag{46}$$

Zeitlin has found many related formulas. For example, Zeitlin ([27], equation 1.14, with $m = 0$ and $n = 0$) found the following interesting formula:

$$w_{kn} = \frac{1}{2^k} \sum_{j=0}^{k} c_j \binom{k}{j} D^{\lfloor j/2 \rfloor} w_n^j v_n^{k-j} \quad \text{where } c_j = \begin{cases} w_0, & \text{if } j \text{ is even,} \\ 2w_1 - Pw_0, & \text{if } j \text{ is odd.} \end{cases} \tag{47}$$

Formula (46) can be converted into a pure formula for w_{kn} be letting $s = 0$ and substituting $u_n = (w_0 w_{n+1} - w_1 w_n)/e$ and $u_{n-1} = (w_1 w_{n+1} - w_2 w_n)/(eQ)$. We get the following.

Theorem 20. If $k \geq 0$, then

$$w_{kn} = \frac{1}{e^k} \sum_{i=0}^{k} \binom{k}{i} (w_0 w_{n+1} - w_1 w_n)^i (w_2 w_n - w_1 w_{n+1})^{k-i} w_i. \tag{48}$$

This can be exanded out as a polynomial in w_n and w_{n+1}. Computer experiments suggest the following result:

Conjecture 21 (The Multiple Argument Formula for w). If k is an integer larger than 1, then

$$w_{kn} = \frac{1}{e^{k-1}} \sum_{i=0}^{k} c_i \binom{k}{i} (-1)^{k-i} w_n^i w_{n+1}^{k-i}, \tag{49}$$

where

$$c_i = \sum_{j=0}^{k-2} \binom{k-2}{j} (-Qw_0)^j w_1^{k-2-j} w_{i-j}. \tag{50}$$

14. Expansion of Products.

Horadam [10] found

$$w_n v_m = w_{n+m} + Q^m w_{n-m}. \tag{51}$$

But we would really like to express $w_n w_m$ as a sum of w's. To do that, we can proceed as follows. Changing m to $m+1$ in equation (51) gives

$$w_n v_{m+1} = w_{n+m+1} + Q^m w_{n-m-1}. \tag{52}$$

But it is easy to show that

$$w_m = \frac{D_1}{D} v_m + \frac{D_2}{D} v_{m+1} \tag{53}$$

where

$$D = P^2 - 4Q, \quad D_1 = P^2 w_0 - 2Q w_0 - P w_1, \quad \text{and} \quad D_2 = 2w_1 - P w_0. \tag{54}$$

Multiplying (51) by D_1/D, multiplying (52) by D_2/D, and adding the results gives us the following theorem.

Theorem 22 (The Product Formula for w). For all integers n and m,

$$w_m w_n = \frac{1}{D}\left[Q^{m+1} D_2 w_{n-(m+1)} + Q^m D_1 w_{n-m} + D_1 w_{n+m} + D_2 w_{n+m+1}\right] \tag{55}$$

where D, D_1, and D_2 are as given in (54).

Applying the symmetrization formula and also expressing $w_{n-(m+1)}$ in terms of w_{n-m} and w_{n-m+1} permits us to obtain another variation of the product formula.

Theorem 23 (Symmetric Product Formula for w). If $w_0 \neq 0$, then for all integers n and m,

$$w_m w_n = \frac{1}{D w_0}\left[eQ^n w_{m-n} - eQ^{m+n} w_{-n-m} + eQ^m w_{n-m} + (D w_0^2 - e) w_{m+n}\right] \tag{56}$$

If $m = n$ in formula (55), we get

$$w_n^2 = \frac{1}{D}\left[2eQ^n + D_1 w_{2n} + D_2 w_{2n+1}\right] \tag{57}$$

which can be used to turn squares into sums. Using formula (56), this can also be written as

$$D w_0 w_n^2 = (D w_0^2 - e) w_{2n} + 2e w_0 Q^n - e w_{-2n} Q^{2n}. \tag{58}$$

Theorem 24. If k is a positive integer, then w_n^k can be expressed in the form

$$(Dw_0)^{k-1} w_n^k = \sum_{i=0}^{k} c_{k,i} Q^{in} w_{(k-2i)n} \tag{59}$$

where $c_{k,i}$ is a polynomial in d, e, and w_0, with integer coefficients, where $d = Dw_0^2$.

Proof: The proof is by induction. The case $k = 2$ is given above in formula (58). Assuming it is true for w_n^k, take the formula for w_n^k and multiply it by $(Dw_0)w_n$. The Symmetric Product Formula then gives the answer in the desired form. □

15. The Power Expansion Formula.

In 1878, Lucas (section XII of [17]) found an explicit formula for w_n in terms of w_0, w_1, P, and Q (see also [25], [12] and [16]):

Theorem 25. For all $n > 0$,

$$w_n = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} P^{n-2k} (-Q)^{k-1} \left[\binom{n-k}{k-1} w_1 P - \binom{n-k-1}{k-1} w_0 Q \right]. \tag{60}$$

16. The Universal Recurrence.

We can solve the system of equations

$$\begin{aligned} w_{n+2} &= Pw_{n+1} - Qw_n \\ w_{n+3} &= Pw_{n+2} - Qw_{n+1} \end{aligned}$$

for P and Q. Thus, any four consecutive terms of the sequence $\langle w_n \rangle$ are enough to determine P and Q. The result is:

$$P = \frac{w_n w_{n+3} - w_{n+1} w_{n+2}}{w_n w_{n+2} - w_{n+1}^2} \qquad \text{and} \qquad Q = \frac{w_{n+1} w_{n+3} - w_{n+2}^2}{w_n w_{n+2} - w_{n+1}^2}. \tag{61}$$

We can substitute these values of P and Q into the identity $w_{n+4} = Pw_{n+3} - Qw_{n+2}$ to arrive at a recurrence for $\langle w_n \rangle$ that does not involve P, Q, w_0, or w_1. The result is the following.

Theorem 26 (The Universal Recurrence). Any second order linear recurrence $\langle w_n \rangle$ with constant coefficients satisfies the recurrence

$$w_{n+4} = \frac{w_{n+2}^3 - 2w_{n+1}w_{n+2}w_{n+3} + w_n w_{n+3}^2}{w_n w_{n+2} - w_{n+1}^2}. \tag{62}$$

We call this the universal recurrence since it is satisfied by any second-order linear recurrence no matter what the coefficients or initial conditions, subject only to the restriction that the denominator should not be 0. [This is equivalent to the condition that $e \neq 0$ and $Q \neq 0$ by formula (24).]

The Universal Recurrence can be written in the form

$$\begin{vmatrix} w_{n+4} & w_{n+3} & w_{n+2} \\ w_{n+3} & w_{n+2} & w_{n+1} \\ w_{n+2} & w_{n+1} & w_n \end{vmatrix} = 0. \tag{63}$$

In this form, the result is due to Casorati.

17. The Recurrence for Multiples.

Zeitlin [24] found the recurrence satisfied by the sequence $\langle w_{kn} \rangle$ where k is a fixed positive integer:

$$w_{kn} = v_k w_{k(n-1)} - Q^k w_{k(n-2)}. \tag{64}$$

This recurrence can be made pure by substituting the value for v_k given by formula (10).

Theorem 27 (The Translation Theorem). Let a be a nonzero integer. Given an identity involving w_n, u_n, and v_n, we can create another valid identity by replacing all occurrences of w_x by w_{x+a}. This operation is called a translation by a.

Proof: Since the original identity is true for a completely arbitrary second-order linear recurrence, $\langle w_n \rangle$, it must be true for the particular second-order linear recurrence $\langle w_{n+a} \rangle$.
□

Theorem 28 (The Dilation Theorem). Let k be a positive integer. Given an identity involving w_n, u_n, and v_n, we can create another valid identity by replacing all occurrences of w_x by w_{kx} provided that we also replace u_x by u_{kx}/u_k, v_x by v_{kx}, Q by Q^k, P by v_k, and e by eu_k^2. This operation is called a dilation by k.

Proof: The sequence $\langle w_{kn}\rangle$ satisfies the second-order linear recurrence given by equation (64). Since the original identity is true for a completely arbitrary second-order linear recurrence, $\langle w_n\rangle$, it must be true for the particular second-order linear recurrence $\langle w_{kn}\rangle$. However, this new recurrence has different parameters; namely $P' = v_k$ and $Q' = Q^k$. If $W_n = w_{kn}$, then the fundamental Lucas sequence $\langle U_n\rangle$ that corresponds to $\langle W_n\rangle$ would satisfy the recurrence $U_n = v_k U_{n-1} - Q^k U_{n-2}$ with initial conditions $U_0 = 0$ and $U_1 = 1$. But the sequence u_{kn} satisfies this recurrence, by (64). To meet the initial conditions, we need only scale it down by a factor of u_k. Thus $U_k = u_{kn}/u_k$. A similar remark holds for the corresponding primordial Lucas sequence $\langle V_n\rangle$.

Thus, if we convert to these new parameters, we should obtain a valid identity. Note that $e = w_0w_2 - w_1^2$ when converted becomes $w_0w_{2k} - w_k^2$ which is equal to eu_k^2 by Theorem 12. $\square$

18. The Recurrence for Powers.

Jarden [15] found the recurrence satisfied by the sequence $\langle w_n^t\rangle$ where t is a fixed positive integer:

$$\sum_{j=0}^{t+1} (-1)^j Q^{j(j-1)/2} \begin{bmatrix} t+1 \\ j \end{bmatrix}_u w_{n-j}^t = 0 \tag{65}$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \frac{u_m u_{m-1} \cdots u_{m-r+1}}{u_1 u_2 \cdots u_r}, \qquad \begin{bmatrix} m \\ 0 \end{bmatrix}_u = 1. \tag{66}$$

See also [9] and [10] for some related identities. Zeitlin [23], [26] has found many other identities involving powers of w's.

19. The Algorithms.

We now summarize the algorithms found earlier in this paper. For the reader's convenience, we repeat some of the earlier formulas (leaving their original formula numbers). All the algorithms listed here have been implemented in Mathematica™, and are available from the author via email.

Algorithm `ConvertToUV` to convert an expression involving w's into one involving u's and v's:

Apply the substitution

$$w_n = \frac{(2w_1 - Pw_0)u_n + w_0v_n}{2}. \tag{4}$$

Algorithm `ConvertToW` to convert expressions involving u's and v's into expressions involving w's:

Apply the identities:

$$\begin{aligned} u_n &= \frac{w_0w_{n+1} - w_1w_n}{e} \\ v_n &= \frac{(Pw_0 - 2w_1)w_{n+1} - (2Qw_0 - Pw_1)w_n}{e}. \end{aligned} \tag{10}$$

Algorithm `WReduce` to remove sums in subscripts:

Repeatedly apply the addition formula

$$w_{n+m} = -\frac{1}{e}\begin{vmatrix} w_0 & w_1 & w_m \\ w_1 & w_2 & w_{m+1} \\ w_n & w_{n+1} & 0 \end{vmatrix}. \tag{13}$$

Algorithm `WNegate` to negate subscripts:

Use the identity

$$w_{-n} = \frac{(w_1^2 - Qw_0^2)w_n + w_0(Pw_0 - 2w_1)w_{n+1}}{eQ^n}. \tag{14}$$

Algorithm `WShift` to change basis:

To convert an expression involving w_n and w_{n+1} into one involving w_{n+a} and w_{n+b}, apply the substitutions

$$\begin{pmatrix} w_n \\ w_{n+1} \end{pmatrix} = \frac{1}{w_{a+1}w_b - w_aw_{b+1}} \begin{pmatrix} w_1 & -w_0 \\ w_2 & -w_1 \end{pmatrix} \begin{pmatrix} w_b & -w_a \\ w_{b+1} & -w_{a+1} \end{pmatrix} \begin{pmatrix} w_{n+a} \\ w_{n+b} \end{pmatrix}. \tag{34}$$

<u>Algorithm `WExpand`</u> to turn products into sums:

Repeatedly apply the substitution

$$w_m w_n = \frac{1}{D}\left[Q^{m+1} D_2 w_{n-m-1} + Q^m D_1 w_{n-m} + D_1 w_{n+m} + D_2 w_{n+m+1}\right] \tag{55}$$

where $D = P^2 - 4Q$, $D_1 = P^2 w_0 - 2Q w_0 - P w_1$, and $D_2 = 2w_1 - P w_0$.

<u>Algorithm `WRemoveP`</u> to remove P in coefficients with terms involving w_c:

If c is a constant, use the identity

$$P^k w_c = \sum_{j=0}^{k} \binom{k}{j} Q^j w_{k+c-2j}. \tag{39}$$

<u>Algorithm `WRemoveQ`</u> to remove Q in coefficients with terms involving w_c:

If c is a constant, use the identity

$$Q^k w_c = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P^{k-j} w_{k+c+j}. \tag{40}$$

<u>Algorithm `RemovePowersOfWPlus1`</u> to remove powers of w_{n+1}:

Use the identity

$$w_{n+1}^2 = P w_n w_{n+1} - Q w_n^2 - e Q^n \tag{67}$$

repeatedly until no w_{n+1} term has an exponent larger than 1. This identity comes from formula (35).

<u>Algorithm `RemovePowersOfW`</u> to remove powers of w_n:

Use the identity

$$w_n^2 = \frac{P w_n w_{n+1} - w_{n+1}^2 - e Q^n}{Q} \tag{68}$$

repeatedly until no w_n term has an exponent larger than 1. This identity comes from formula (35).

Algorithm `RemovePowersOfQ` to remove variable powers of Q:

To remove any expressions of the form Q^{an+b} from an expression, where n is a variable and a and b are independent of n with $a \neq 0$, write Q^{an+b} as $Q^b(Q^n)^a$ if $a > 0$ and as $Q^b(Q^{-n})^{-a}$ if $a < 0$. Then replace $Q^{\pm n}$ by the substitution

$$Q^{\pm n} = \frac{Pw_{\pm n}w_{\pm n+1} - Qw_{\pm n}^2 - w_{\pm n+1}^2}{e} \tag{69}$$

which come from formula (35). If $a < 0$, we cannot in general replace Q^{an} by any polynomial in the w's with subscripts consisting only of positive multiples of n. However, if Q happens to be a root of unity, then simplification is possible. The cases $Q = -1$ and $Q = 1$ frequently occur and are of this form. Let m be the smallest positive integer such that $Q^m = 1$. Write Q^{an+b} as Q^bQ^{an}. Let b be the residue of a modulo m, i.e. the positive integer such that $0 \leq b < m$ and $b \equiv a \pmod{m}$. Then $Q^a = Q^b$, so we can replace Q^{an} by Q^{bn} with $b \geq 0$. If $b > 0$, we proceed as in the previous case.

Definition. A w-polynomial is any polynomial $f(x_1, x_2, \ldots, x_r)$ with constant coefficients where each x_i is of the form w_x, u_x, v_x, or Q^x, with each x of the form $a_1n_1 + a_2n_2 + \cdots + a_kn_k + b$, where b and the a_i are integer constants and the n_i are variables. For purposes of this definition, the quantities P, Q, w_0, and w_1 are to be considered constants.

Algorithm `WSimplify` to convert an expression to canonical form:

INPUT: A w-polynomial.

OUTPUT: Its "canonical form". Two expressions that are identical will have the same canonical form. In particular, an expression is identically 0 if and only if its canonical form is 0.

STEP 1: [Convert to w.] If any expression of the form u_x or v_x occurs, apply algorithm `ConvertToW` to remove it.

STEP 2: [Remove variable sums in subscripts.] If any expression of the form $a_1n_1 + a_2n_2$ occurs in a subscript, apply algorithm `WReduce` to remove such sums. Treat $a_1n_1 - a_2n_2$ as $a_1n_1 + (-a_2)n_2$.

STEP 3: [Make multipliers positive.] All subscripts are now of the form $an + b$ where a and b are integers and n is a variable. For any term in which the multiplier a is negative, apply algorithm `WNegate`.

STEP 4: [Remove multipliers.] All subscripts are now of the form $an + b$ where a is a nonnegative integer, b is an integer, and n is a variable. If $a > 1$, write $an + b$ as $n + n + \cdots + n + b$ with a copies of n and then apply algorithm `WReduce` repeatedly until all these subscripts are of the form $n + c$ where c is an integer.

STEP 5: [Remove constants in subscripts.] If any expression of the form $n + b$ with $b \neq 0$ and $b \neq 1$ occurs in a subscript, apply algorithm `WReduce` to remove such sums.

STEP 6: [Remove powers of w_{n+1}.] If any term involves an expression of the form w_{n+1}^k where $k > 1$ and n is a variable, apply algorithm `Remove-PowersOfWPlus1` to leave only linear terms in w_{n+1}.

STEP 7: [Evaluate constants.] If any term involves an expression of the form w_c^k where c is an integer constant, replace w_c by its numerical equivalent. If the symbols D or e occur, replace them by their equivalent values from formula (7).

STEP 8: [Simplify Powers of Q.] If Q is a primitive m-th root of unity, then replace all constants appearing in an exponent with base Q by their residues modulo m.

The canonical form is a polynomial $f(x_1, x_2, \ldots, x_r)$ with constant coefficients where each x_i is of the form w_{n_i}, w_{n_i+1}, or $Q^{\pm n_i}$, where the n_i are variables, and the degree of each w_{n_i+1} term is 0 or 1. If Q is a root of unity, then no exponent with base Q is negative.

Alternatively, to prove an identity, you can apply algorithm `LucasSimplify` and show that the resulting canonical form is 0.

References

[1] Irving Adler, "Sequences with a Characteristic Number", *The Fibonacci Quarterly* **9.2**(1971)147–162.

[2] Brother U. Alfred, "On the Ordering of the Fibonacci Sequence", *The Fibonacci Quarterly* **1.4**(1963)43–466.

[3] J. P. M. Binet, "Mémoire sur l'intégration des équations linéaires aux différences finies d'un ordre quelconque à coefficients variables", *Comptes Rendus des Séances de l'Académie des Sciences (Paris)* **17**(1843)561–565.

[4] Eugène-Charles Catalan, "Sur la série de Lamé", *Mémoires de la Société Royale des Sciences de Liège* (series 2) **13**(1886)319–321.

[5] Ernest Cesàro, "Solution to Problem 570 (Proposed by Gelin)", *Nouvelle Correspondance Mathématique* **6**(1880)423–424.

[6] Maurice d'Ocagne, "Sur une Suite Récurrente", *Bulletin de la Société Mathématique de France* **14**(1885-1886)20–41.

[7] Dustan Everman, A. E. Danese, and K. Venkannayah, "Problem E1396", *The American Mathematical Monthly* **67**(1960)81–82.

[8] Peter Gilbert, personal communication.

[9] V. E. Hoggatt, Jr. and D. A. Lind, "A Power Identity for Second-Order Recurrent Sequences", *The Fibonacci Quarterly* **4.3**(1966)274–282.

[10] A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers", *The Fibonacci Quarterly* **3.3**(1965)161–176.

[11] A. F. Horadam, "Generating Function for Powers of a Certain Generalized Sequence of Numbers", *Duke Math. Journal* **32**(1965)437–446.

[12] A. F. Horadam, "Tschebyscheff and Other Functions Associated with the Sequence $\{w_n(a,b;p,q)\}$", *The Fibonacci Quarterly* **7.1**(1969)14–22.

[13] A. F. Horadam and A. G. Shannon, "Generalization of Identities of Catalan and Others", *Portugaliae Mathematica* **44**(1987)137–148.

[14] Dov Jarden and Th. Motzkin, "A Multiplicatory Formula for the General Recurring Sequence of Order 2", *The Mathematics Student* **13**(1945)61–63.

[15] Dov Jarden, "The Product of Sequences with a Common Linear Recursion Formula of Order 2" in Recurring Sequences, second edition (edited by Dov Jarden), Riveon Lematematika. Jerusalem, Israel: 1966. pp. 30–33.

[16] Jin-Zai Lee and Jia-Sheng Lee, "Some Properties of the Sequence $\{W_n(a,b;p,q)\}$", *The Fibonacci Quarterly* **25.3**(1987)268–278, 283.

[17] Edouard Lucas, "Théorie des Fonctions Numériques Simplement Périodiques", *American Journal of Math.* **1**(1878)184–240, 289–321.

[18] R. S. Melham and A. G. Shannon, "Some Congruence Properties of Generalized Second-Order Integer Sequences", *The Fibonacci Quarterly* **32.5**(1994)424–428.

[19] Stanley Rabinowitz, "Algorithmic Manipulation of Fibonacci Identities" in Applications of Fibonacci Numbers, volume 6 (edited by G. E. Bergum et al.), Kluwer Academic Publishers, Dordrecht, The Netherlands: 1996. pp. 389–408.

[20] A. G. Shannon and A. F. Horadam, "Special Recurrence Relations Associated with the Sequence $\{w_n(a,b;p,q)\}$", *The Fibonacci Quarterly* **17.4**(1979)294–299.

[21] Robert Simson, "An Explication of an Obscure Passage in Albrecht Girard's Commentary upon Simon Stevin's Works", *Philosophical Transactions of the Royal Society of London* (series 1) **48**(1753)368–376.

[22] Alberto Tagiuri, "Di alcune successioni ricorrenti a termini interi e positivi", *Periodico di Matematica* **16**(1901)1–12.

[23] David Zeitlin, "Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$", *The Fibonacci Quarterly* **3.4**(1965)241–256.

[24] David Zeitlin, "On Summation Formulas and Identities for Fibonacci Numbers", *The Fibonacci Quarterly* **5.1**(1967)1–43.

[25] David Zeitlin, "On Convoluted Numbers and Sums", *The American Mathematical Monthly* **74**(1967)235–246.

[26] David Zeitlin, "On Determinants whose Elements are Products of Recursive Sequences", *The Fibonacci Quarterly* **8.4**(1970)350–359.

[27] David Zeitlin, "General Identities for Recurrent Sequences of Order Two", *The Fibonacci Quarterly* **9.4**(1971)357–388, 421–422.

AMS Classification Numbers: 11Y16, 11B37.