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Convex Lattice Polytopes

Section 1
Preliminary Material

Introduction. In this dissertation, we will investigate inequalities involving lattice polytopes and will study related properties of convex polytopes in E^n. Hammer has published a book [50] listing unsolved problems concerning lattice points. This thesis was inspired by Hammer's book and by the work of Scott (see bibliography). Lattice point problems arise in many fields of mathematics, including Convexity, Number Theory, Combinatorics, Algebraic Geometry, Combinatorial Geometry, and Geometry of Numbers.

Many problems concerning lattice points are easy to state, yet hard to prove. An example is a lemma of Kempf, Knudsen, Mumford, and Saint-Donat, in their volume, Toroidal Embeddings [63]. It states that if P is a lattice polytope in E^n, then there exists an integer ν and a subdivision of P into finitely many simplices, P_j, each of volume $1/n!\nu^n$, such that, for all j, every vertex of P_j lies in $(1/\nu)Z^n$. The shortest known proof of this easy-to-state lemma is 55 pages in length. The interested reader can find the proof in [63].

In this section, we give the basic definitions and known results that the reader will need. We start by specifying the notation to be used and recalling some basic definitions.

We use E^n to denote Euclidean n-dimensional space. A *lattice point* in E^n is a point with integer coordinates (in a Euclidean rectangular coordinate system). A *lattice polytope* (or lattice polyhedron) is a polytope with non-zero volume in E^n all of whose vertices are lattice points. A *lattice polygon* is a simple polygon in the plane (with non-zero area) all of whose vertices are lattice points.

Notation. By TRIANG(p, h) we denote the triangle whose vertices are $(0, 0)$, $(p, 0)$, and $(0, h)$. By TRAP(p, q, h) we denote the trapezoid whose vertices are $(0, 0)$, $(p, 0)$, $(0, h)$, and (q, h).

Let Z denote the set of integers, Z^+ denote the set of positive integers, and Z^n denote the set of lattice points in E^n.

Let $\lfloor x \rfloor$ denote the *floor* of x, the largest integer not larger than x and let $\lceil x \rceil$, denote the *ceiling* of x, the smallest integer not smaller than x. We use $\gcd(m, n)$ to denote the greatest common divisor of the integers m and n and $\operatorname{lcm}(m, n)$ to denote the least common multiple of the integers m and n. We let card(S) denote the number of points in the set S.

Definition. If A and B are two lattice points in E^n, the *lattice length* of segment AB is defined to be one less than the number of lattice points on the segment AB; i.e., the lattice length of AB is card$(AB \cap Z^n) - 1$.

We note that lattice length coincides with Euclidean length for segments parallel to one of the coordinate axes.

Definition. A *lattice line* is a line that contain two lattice points. A *lattice segment* is the line segment joining two lattice points. A *rational point* is a point with rational coordinates.

We note that if two distinct lattice lines meet in a point then their point of intersection must be a rational point.

We now recall some definitions and facts concerning unimodular transformations. A *unimodular transformation* of E^n is a linear transformation that preserves volume. We recall that the matrix representation for such a transformation has a determinant of ± 1.

For example, the general form of a unimodular transformation in the plane is:

$$x' = ax + by$$

$$y' = cx + dy$$

$$|ad - bc| = 1.$$

An integral transformation of E^n is a linear transformation of E^n that maps Z^n onto Z^n.

It can be shown that an integral unimodular transformation of E^n is a unimodular transformation whose matrix representation (with respect to the standard basis of E^n) consists of integer entries.

An *integral translation* of E^n is a translation of E^n that maps Z^n onto Z^n.

An *affine transformation* is the composition of a linear transformation and a translation. We note that this is a continuous transformation of E^n that preserves collinearity. A *unimodular affine tranformation* is the composition of a unimodular transformation and a translation. An intergal unimodular affine transformation is thus the composition of an integral unimodular transformation and an integral translation. (This transformation is sometimes referred to as an *equiaffine transformation*). An integral unimodular affine transformation is sometimes called a *lattice equivalence.* A *congruence* is a distance preserving transformation of E^n and a *lattice congruence* is a congruence that maps Z^n onto Z^n.

A *shear* is an integral unimodular transformation that leaves all the points on a given line fixed. It is also referred to as a shear about this line.

We note that in E^2, a shear about the x-axis is given by the equations

$$x' = x + ky$$

$$y' = y$$

$$k \in Z.$$

Such a shear is said to have magnitude k, and if $k = 1$, the shear is called a *unit shear.*

Definition. Two lattice polytopes are said to be *lattice equivalent* if one can be transformed into the other by a lattice equivalence. Two lattice polytopes are said to be *lattice congruent* if one can be transformed into the other by a lattice congruence.

We note that if two polytopes are lattice congruent, then they are congruent and lattice equivalent.

Remark. According to Klein's Erlangen Program, the various geometries can be characterized by the groups of transformations preserving the properties of interest in that geometry. For example, congruence transformations are fundamental in traditional Euclidean geometry. That is because two figures that are congruent have exactly the same set of properties which are considered important in the study of Euclidean geometry. However, in the study of lattice polytopes, this is not the case. Two congruent figures may have different lattice properties. For example, the square with edge joining the origin to $(5, 0)$ is congruent to the square with edge joining the origin to $(3,4)$. These two squares have a different number of interior lattice points. They also have a different number of lattice points on their boundary. Since we will be interested in the number of lattice points in the interior of a figure, we cannot treat two figures alike because they are congruent. It will turn out that lattice equivalence will be good enough for many of our theorems. Thus, it is the integral unimodular affine transformation that is of fundamental importance when studying lattice polytopes since we will show (in section 2) that this group of transformations preserves the properties we are most interested in. The stronger property of lattice congruence will occasionally be necessary.

We now recall some basic notation and definitions.

If K is a convex set, K° will denote the interior of K, and ∂K will denote the boundary of K. We will use $G(K)$ to denote the number of lattice points in K; i.e., $G(K) = \mathrm{card}(K \cap Z^n)$. We will use $g(K)$ to denote the number of lattice points in the interior of the set K; i.e., $g(K) = G(K^\circ)$.

A *convex figure* in E^n is a compact convex subset of E^n. A *convex body* is a convex figure with interior points. We use S^{n-1} to denote the unit sphere in E^n centered at the origin. A *ball* is the convex hull of a sphere.

We recall that for a convex body, K, and a unit vector $\mathbf{u}$, the *width of* K *in the direction* $\mathbf{u}$, is the distance between the two hyperplanes orthogonal to $\mathbf{u}$ which support K, denoted by $b_K(u)$. Recall that $\max\{b_K(u)|u \in S^{n-1}\}$ is the *diameter* of K, which we denote by $D(K)$; while $\min\{b_K(u)|u \in S^{n-1}\}$ is called the *minimal width* of K, which we denote by $w(K)$. The minimal width is sometimes called the breadth or thickness; we will also refer to it simply as the *width* of K.

The *inradius* of a convex body, K, is the radius of a largest ball contained within K, while the *circumradius* is the radius of the smallest ball containing K. The existence of these balls can be shown by appealing to the Blaschke Selection Theorem.

Section 1.2.
Elementary properties of lattice points and lattice polygons.

The following trivial property of a lattice segment will be needed.

Proposition 1.2.1. *If $A = (x_1, x_2, \ldots, x_n)$ and $B = (x'_1, x'_2, \ldots, x'_n)$ are two lattice points in E^n, then*

$$g(AB) = \gcd(x'_1 - x_1, x'_2 - x_2, \ldots, x'_n - x_n) - 1.$$

In other words, the lattice length of AB is $\gcd(x'_1 - x_1, x'_2 - x_2, \ldots, x'_n - x_n)$.

Proof. Let $d = \gcd(x'_1 - x_1, x'_2 - x_2, \ldots, x'_n - x_n)$ and consider the $d + 1$ points,

$$(\frac{x'_1 - x_1}{d}k, \frac{x'_2 - x_2}{d}k, \ldots, \frac{x'_n - x_n}{d}k)$$

as k varies from 0 to d. These points are all lattice points, so there are at least as many lattice points on AB as the proposition claims. If there were some other lattice point on this segment, consider the two lattice points of the above sequence nearest to this point. It is clear from examining the differences between their corresponding coordinates that this would yield a smaller number than d dividing all $x'_i - x_i$, yielding a contradiction. $\square$

Pick's Formula. *Let K be a lattice polygon with area A, g interior lattice points, and b lattice points on its boundary. Then*

$$A = \frac{b}{2} + g - 1.$$

This is a well-known result. For a proof, consult Coxeter [24]. See section 15 for generalizations to higher dimensions.

Proposition 1.2.2. *Alternate formulations:*

$$A = \frac{G}{2} + \frac{g}{2} - 1$$

$$A = G - \frac{b}{2} - 1$$

$$G = 2A - g + 2.$$

These follow trivially from the fact that $G = b + g$.

Section 1.3.
General inequalities for convex sets in the plane.

We will need to refer to various inequalities for convex bodies in the plane. For completeness and quick reference, we list them here. We begin with some known inequalities involving two parameters.

Inequalities 1.3.1. *If a convex figure in the plane has area A, diameter D, inradius r, circumradius R, perimeter P, (minimal) width w and g interior lattice points, then*

a. $w \leq D$.
b. $\pi w \leq P$.
c. $P \leq \pi D$.
d. $2D \leq P$.
e. $4A \leq \pi D^2$.
f. $4\pi A \leq P^2$ (Isoperimetric Inequality).
g. $R \leq D/\sqrt{3}$ (Jung's Theorem).
h. $w^2 \leq A\sqrt{3}$.
i. $w \leq 3r$ (Blaschke's Theorem).

Equality holds when and only when K is a curve of constant width (a-c); line sement (d); circle (e-f); or equilateral triangle (g-i) respectively.

References.
a-g. Sholander [114].
h-i. Eggleston [29].

The following are some known inequalities involving more than two parameters.

Inequalities 1.3.2. *If a convex figure in the plane has area A, diameter D, inradius r, circumradius R, perimeter P, (minimal) width w and g interior lattice points, then*

a. $(w - 2r)D \leq 2\sqrt{3}r^2$.
b. $(w - 2r)A \leq w^2 r/\sqrt{3}$.
c. $Pr \leq 2A$.
d. $A + \pi r^2 \leq rR$.
e. $A + \pi R^2 \leq RP$.
f. $A < wD \leq 2A$.
g. $3wD - \sqrt{3}D^2 \leq 2A$.
h. $\pi w^2 - \sqrt{3}D^2 \leq 2A$.
i. $2\sqrt{D^2 - w^2} + 2w \arcsin \frac{w}{D} \leq P$.
j. $P \leq 2\sqrt{D^2 - w^2} + 2D \arcsin \frac{w}{D}$.
k. $2A \leq w\sqrt{D^2 - w^2} + D^2 \arcsin \frac{w}{D}$.
l. $4A \leq 2wP - \pi w^2$.
m. $8\phi A \leq P(P - 2D\cos\phi)$ where ϕ satisfies $2\phi D = P \sin\phi$.
n. $4\sqrt{3}w^2 - Pw \leq 6A$.
o. $Pw - 2w^2/\sqrt{3} \leq 4A$.
p. $Pw \leq 6A$.
q. $(R - r)^2 \leq P^2 - 4\pi A$.
r. $4A \leq PD$.
s. $(P - 2D)\sqrt{3}D \leq 4A$.
t. $A \leq r(P - \pi r)$.
u. $R(P - 4R) \leq A$.

References.

a. Scott [103].
b. Scott [104].
c. Scott [107].
d-e. Eggleston [29], page 110.
f-p. Sholander [114].
q. Eggleston [29], page 108.
r-u. Bonneson and Fenchel [16], page 81.

The following are some known inequalities involving g.

Inequalities 1.3.3. *If a convex body in the plane has g interior lattice points, area A, and perimeter P, then*

a. $g > A - P/2$.
b. $g < A + P/2 + 1$.

Reference.

a-b. Nosazewska [77].

Section 1.4.
General inequalities for convex polygons in the plane.

Inequalities 1.4.1. *For a convex n-gon in E^2 with circumradius R, area A, inradius r, and perimeter P, the following inequalities are known.*

a.
$$A \leq 2nR^2 \cos\frac{\pi}{n}\tan\frac{\pi}{2n}$$

b.
$$nr^2\tan\frac{\pi}{n} \leq A \leq \frac{1}{2}nR^2\sin\frac{2\pi}{n}$$

c.
$$2nr\tan\frac{\pi}{n} \leq P \leq 2nR\sin\frac{\pi}{n}$$

d.
$$\frac{P^2}{A} \geq 4n\tan\frac{\pi}{n}$$

with equality if and only if the n-gon is regular.

Reference. Bottema [17] and Fejes Tóth [34], p. 153.

Section 1.5.
General inequalities for convex polytopes in E^n.

Let $V_n(r)$ denote the volume of the sphere of radius r and $A_n(r)$ denote the surface area of the sphere of radius r. It is well-known that $V_n(r) = \pi^{n/2}r^n/\frac{n}{2}\Gamma(\frac{n}{2})$ and $A_n(r) = 2\pi^{n/2}r^{n-1}/\Gamma(n/2)$ where Γ denotes the Gamma function. (If m is an integer, $\Gamma(m) = (m-1)!$ and $\Gamma(m+\frac{1}{2}) = (2m)!\sqrt{\pi}/m!2^{2m}$.)

Inequality 1.4.2. *If a convex body in E^n has g interior lattice points, volume V and surface area F, then $g > V - F/2$ where the factor $1/2$ is best possible.*

Reference. Bokowski, Hadwiger, und Wills [12].

Inequalities 1.4.3. *If a convex body in E^n has volume V, surface area F, diameter D, inradius r, circumradius R, and (minimal) width w, then*

a. $2w^n \leq n!V\sqrt{3}$.
b. $2(n+1)R^2 \leq nD^2$.
c. $r \geq \begin{cases} \frac{w}{2\sqrt{n}} & n \text{ odd} \\ \frac{w\sqrt{n+2}}{2(n+1)} & n \text{ even.} \end{cases}$
d. $V_n(r) \leq V \leq V_n(D/2) \leq V_n(R)$.
e. $A \leq A_n(D/2) \leq A_n(R)$.
f. $r \leq R$.
g. $2r \leq D$.
h. $D \leq 2R$.
i. $2r \leq w$.
j. $w \leq 2R$.
k. $A_n(r) \leq A_n(w/2) \leq A$.
l. $w \leq D$.

Reference. Eggleston [29], pages 104-111.

We collect for quick reference, the following "best possible" inequalities.

Inequalities 1.4.4. *If a convex body in E^n has g interior lattice points, volume V, surface area F, diameter D, inradius r, circumradius R, and (minimal) width w, then*

a. $0 \leq V \leq \frac{1}{2n}\left(\frac{1}{\sqrt{\pi}}\Gamma(\frac{n}{2})A\right)^{n/(n-1)}$.
b. $0 \leq V \leq \frac{2}{n\Gamma(n/2)}(\frac{\sqrt{\pi}}{2}D)^n$.
c. $\frac{\sqrt{\pi}}{\Gamma(n/2)}(2nV)^{(n-1)/n} \leq A < \infty$.
d. $0 \leq A \leq \frac{2\sqrt{\pi}}{\Gamma(n/2)}(\frac{\sqrt{\pi}}{2}D)^{n-1}$.
e. $\frac{2}{\sqrt{\pi}}\sqrt[n]{\frac{n}{2}\Gamma(\frac{n}{2})V} \leq D < \infty$.
f. $\frac{2}{\sqrt{\pi}}\sqrt[n-1]{\frac{1}{2\sqrt{\pi}}\Gamma(\frac{n}{2})A} \leq D < \infty$.
g. $0 \leq r \leq \frac{1}{\sqrt{\pi}}\sqrt[n]{\frac{n}{2}\Gamma(\frac{n}{2})V}$.
h. $0 \leq r \leq \frac{1}{\sqrt{\pi}}\sqrt[n-1]{\frac{1}{2\sqrt{\pi}}\Gamma(\frac{n}{2})A}$.
i. $0 \leq r \leq \frac{1}{2}D$.
j. $0 \leq w \leq \sqrt[n]{\frac{\sqrt{3}}{2}n!V}$.
k. $0 \leq w \leq \frac{2}{\sqrt{\pi}}\sqrt[n-1]{\frac{1}{2\sqrt{\pi}}\Gamma(\frac{n}{2})A}$.
l. $0 \leq w \leq D$.
m. $\frac{1}{\sqrt{\pi}}\sqrt[n]{\frac{n}{2}\Gamma(\frac{n}{2})V} \leq R < \infty$.
n. $\sqrt[n-1]{\frac{\Gamma(n/2)}{2\pi^{n/2}}A} \leq R < \infty$.
o. $\frac{1}{2}D \leq R \leq D\sqrt{\frac{n}{2(n+1)}}$.
p. $\frac{2}{n\Gamma(n/2)}(r\sqrt{\pi})^n \leq V < \infty$.
q. $\frac{2}{n!\sqrt{3}}w^n \leq V < \infty$.
r. $0 \leq V \leq \frac{2}{n\Gamma(n/2)}(R\sqrt{\pi})^n$.
s. $\frac{2\sqrt{\pi}}{\Gamma(n/2)}(r\sqrt{\pi})^{n-1} \leq A < \infty$.
t. $\frac{2\sqrt{\pi}}{\Gamma(n/2)}(\frac{\sqrt{\pi}}{2}w)^{n-1} \leq A < \infty$.
u. $0 \leq A \leq \frac{2\pi^{n/2}}{\Gamma(n/2)}R^{n-1}$.
v. $2r \leq D < \infty$.

w. $w \le D < \infty$.

x. $R\sqrt{\frac{2(n+1)}{n}} \le D \le 2R$.

y. $\frac{\sqrt{n+1+(-1)^n}}{2n+1+(-1)^n} w \le r \le \frac{1}{2}w$.

z. $0 \le r \le R$.

aa. $2r \le w \le \frac{2n+1+(-1)^n}{\sqrt{n+1+(-1)^n}} r$.

bb. $0 \le w \le 2R$.

cc. $r \le R < \infty$.

dd. $\frac{1}{2}w \le R < \infty$.

When we write an inequality in the form $f(p) \le q < \infty$, we mean that q is not bounded above by any function of p.

Section 1.6.
Minkowski's Theorem.

No discussion of lattice points in convex sets would be complete without mention of the following fundamental theorem of Minkowsky. For proofs of these results, consult any book on the Geometry of Numbers, such as Lekkerkerker [68] or Cassels [19].

Minkowski's Theorem. *Let K be a convex body centrally symmetric about the origin in E^n. If $V(K) \geq 2^n$, then K contains a non-zero lattice point.*

One can consider lattices more general than Z^n. We can consider a lattice Λ, in E^n, generated by n independent vectors emenating from the origin. The determinant of this lattice is the determinant of the matrix whose rows are the coordinate vectors corresponding to the endpoints of these vectors. The determinant of this lattice is denoted by $d(\Lambda)$.

We recall the following definitions.

Definitions. If K is a convex body, then a lattice Λ is said to be *K-admissible* if no point of Λ (other than perhaps the origin, O) lies in the interior of K. The *critical determinant*, $\Delta(K)$ of K is defined as the infimum of $d(\Lambda)$ taken over all K-admissible lattices. A lattice which is admissible for K and for which $d(\Lambda) = \Delta(K)$ is called a *critical lattice* for K. Let $V(K)$ denote the volume of K.

Using this notation, we can rephrase Minkowski's Theorem in a more general setting.

Minkowski's Theorem. *If K is a centrally symmetric convex body in E^n centered at the origin, O, then*

$$\frac{1}{2^n} \leq \frac{\Delta(K)}{V(K)}.$$

An upper bound to $\Delta(K)/V(K)$ is also known.

Minkowski-Hlawka Thoerem. *If K is a centrally symmetric convex body in E^n centered at the origin, O, then*

$$\frac{\Delta(K)}{V(K)} \leq \frac{1}{2\zeta(n)},$$

where $\zeta(n)$ is the Riemann-Zeta function,

$$\zeta(n) = 1 + 2^{-n} + 3^{-n} + \cdots \quad .$$

Section 2
Properties of Integral Unimodular Transformations

In this section, we establish properties of integral unimodular transformations that will be needed in later sections.

Proposition 2.1. *If f is an integral unimodular transformation and K is a convex body in E^n, then $G(f(K)) = G(K)$.*

In other words, integral unimodular transformations preserve the number of lattice points in sets.

Proof. The integral unimodular transformation can be expressed as a set of n linear equations relating the old coordinates to the new coordinates. The ith equation has the new coordinate x_i' on the left side of the equal sign. The right side of the equal sign is an integral linear combination of all the original coordinates, x_j. By Kramer's Rule, we can invert this system of equations, to obtain a new system of linear equations with the old coordinates on the left and the new coordinates on the right. Each expression on the right hand side is the quotient of two determinants. The determinant in the denominator is the same for all n equations and consists of the coefficient matrix of the original set of equations. Since the transformation is unimodular, this determinant has value ± 1. The numerator in each case is a determinant, all of whose entries are integers. Hence, we see that this inverse transformation maps lattice points into lattice points. Since the original transformation also has this property, we see that lattice points get mapped into lattice points; and lattice points can only be obtained from other lattice points. $\square$

Corollary. *Integral unimodular transformations preserve lattice length.*

Proposition 2.2. *If f is an integral unimodular transformation and K is a convex body in E^n, then $g(f(K)) = g(K)$.*

In other words, integral unimodular transformations preserve lattice points in the interior of sets.

Proof. Apply Proposition 2.1 to the interior of K and note that a homeomorphism maps the interior of a set into the interior of the image. A unimodular transformation is a homeomorphism since the vector space involved is finite-dimensional and hence all linear transformations are continuous. $\square$

The x-axis Lemma. *Let AB be a side of a convex lattice polygon K in E^2. Let m be the lattice length of AB. Then there is an integral unimodular affine transformation that maps A into the origin, maps B into the point $(m, 0)$ on the positive x-axis, and maps all the other vertices of K into points above the x-axis.*

Proof. First translate the point A to the origin. This is an integer translation. Let B be transformed into the point (p, q) by this translation. We seek integers a, b, c, d such that $ad - bc = 1$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms B into a point on the x-axis. In other words, we want

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

for some positive integer x. Let $r = \gcd(p, q)$. Then we merely need pick $c = q/r$ and $d = -p/r$. This makes $cp + dq = 0$ and a and b need be chosen so that $ad - bc = 1$. Such positive integers can be found by a well-known theorem from number theory since q/r and p/r are relatively prime. $\square$

Proposition 2.3. *The product of two integral unimodular transformations is another integral unimodular transformation.*

This follows from the fact that the determinant of the product of two matrices is the product of the determinants of the two matrices.

Proposition 2.4. *Let K be any convex body in the plane. Then by a suitable integral unimodular transformation, we can make D (the Diameter) arbitrarily large and w (the width) arbitrarily small.*

Proof. Without loss of generality, we assume that the origin lies in K. Since K is a body (contains an interior), it contains at least one other point, say P. Clearly we can transform P to be as far away from the origin as we want by making a, b, c, and d large, where a, b, c, and d are as defined in the proof of the x-axis Lemma. This can be done by picking a and b to be large and relatively prime, and then using the well-known number theory algorithm to find c and d so that $ad - bc = 1$. Thus D can be made arbitrarily large. By Proposition 1.3.2f, $wD \leq 2A$ and since area is preserved, this shows that we can also make w arbitrarily small. □

Proposition 2.5. *Any convex lattice polygon is equivalent to a convex lattice polygon contained wholly within the first quadrant (including the axes) and with one vertex at the origin.*

Proof. Apply the x-axis Lemma to map one edge onto the x-axis with one vertex at the origin. Since the polygon is convex, it lies entirely on one side of this edge. If that side is below the x-axis, apply a reflection about the x-axis. Then, for each vertex in quadrant 2, apply the appropriate shear (about the x-axis) to move that vertex into quadrant 1. □

We will also need later a lemma, first given by Scott, and we refer the reader to [101] for the proof.

Lemma 2.6. *Let AB, CD be segments lying along the x-axis, having integral endpoints, and lengths p, q, respectively. Let h be a positive integer such that $h > p+q$. Then there exists lattice points P and Q lying on AB, CD, respectively, such that $PQ = mh + u$ where m is a non-negative integer and $|u| \leq (h - p - q)/2$.*

Section 3
Convex lattice polytopes containing exactly g interior lattice points

In this section we will exhibit several classes of convex lattice polytopes containing exactly g interior lattice points. These will be useful later.

Proposition 3.1. *Let g be any non-negative integer. Then TRIANG$(2g+2,2)$ contains exactly g interior lattice points.*

Proof. The base contains $2g+3$ lattice points, and each of the other sides contain 3 lattice points, so the total number of lattice points on the boundary is $2g+6$. The area of the triangle is $2g+2$, so by Pick's Formula, the number of interior lattice points is $A-b/2+1 = 2g+2-(g+3)+1 = g$. □

Notation. We use WEDGE(g) to denote the triangle whose vertices are at $(0,0)$, $(1,2)$, and $(g+1,1)$.

Proposition 3.2. *Let g be any non-negative integer. Then WEDGE(g) contains exactly g interior lattice points.*

Proof. Clearly, the only interior lattice points are of the form $(k,1)$, as k ranges from 1 to g. There are exactly g such lattice points. □

Proposition 3.3. *Let g be any non-negative integer. Then there is a simplex in E^n containing exactly g interior lattice points.*

Proof. Let $A_i, i=1,2,\ldots,n$ be the point in E^n whose ith coordinate is 1 and whose other coordinates are all 0. Let A_0 be the point whose nth coordinate is 0 and whose other coordinates are all -1. These $n+1$ points form a simplex with no interior lattice points. If we now change the nth coordinate of A_n from 1 to $g+1$, we wind up with a simplex containing exactly g interior lattice points. □

Definition. A lattice polytope is said to be *lean* if the vertices are the only lattice points on the boundary of the polytope.

Proposition 3.4. *For any non-negative integer g, there is a lean triangle containing precisely g interior lattice points.*

This follows from the fact that WEDGE(g) is lean.

Proposition 3.5. *For any non-negative integer A, there is a lean lattice triangle with area $A+1/2$.*

Proof. Applying Pick's Formula to the lean triangle found in Proposition 3.4 shows that the area of that triangle is $b/2+g-1 = g+1/2$. This is thus the desired triangle. □

Other results about lean polygons will be found in section 7. For example, it will be shown that for any integer $v \geq 3$, we can find a convex lean polygon with precisely v vertices.

We note that TRAP$(g+2, g+2, 2)$ is a rectangle containing exactly g interior lattice points.

The Enclosed Square Lemma. *Let g be a positive integer. Then there is a convex lattice polygon with $v \leq 8$ containing exactly g interior lattice points and bounded by a square of side $\lceil\sqrt{g}\rceil+1$.*

Proof. Suppose that g is between n^2+1 and $(n+1)^2$ inclusive, so that $\lceil\sqrt{g}\rceil = n+1$.

If $g = (n+1)^2$, then we may take the square with side length $n+2$ whose sides are parallel to the axes. This contains exactly g interior lattice points.

Keeping the outer square $ABCD$ fixed, we wish to show that we can remove interior lattice points by making the enclosing convex lattice polygon smaller. The number of lattice points we need remove varies from 0 to $2n$ since $(n+1)^2 - 2n = n^2 + 1$. It suffices to show that we can remove anywhere from 0 to n lattice points along edges AB and BC or near vertex B, not including the interior lattice points closest to vertices A and C. For then we can apply the same process around vertex D, along edges DA and DC.

Consider point B to be the origin. To remove one lattice point, move the vertex at $(0,0)$ to $(1,1)$. To remove two lattice points, choose vertices at $(2,0)$ and $(0,4)$ instead. To remove three lattice points, choose as vertices $(3,0)$ and $(0,3)$. To remove four lattice points, choose $(2,1)$ and $(0,5)$ as vertices. To remove five lattice points, choose $(2,1)$ and $(0,7)$ as vertices. To remove six lattice points, choose $(4,0)$ and $(0,4)$ as lattice points. (Note: this assumes $n > 6$. Smaller values of n can easily be handled as special cases.)

Finally, to remove from 7 to n lattice points, choose $(2,2)$ as a vertex and then we show how to remove $\lfloor n/2 \rfloor$ lattice points from along edge BC. Since we can then do the same along edge AB, this allows us to remove a total of up to n lattice points. To remove 1 additional lattice point, choose $(4,0)$ as a new vertex. To remove 2 additional lattice points, choose $(6,0)$ as a new vertex instead. We can continue to do this until vertex D or the lattice point just to its left is reached. This allows a total of $\lfloor n/2 \rfloor$ additional lattice points to be removed. $\square$

Section 4
An algorithm for searching lattice polygons

There are many examples of mathematical theorems whose proof requires the examination of a large number of cases. If an algorithm is known for verifying these cases, then a computer can be used to establish the validity of these cases. In the study of lattice polytopes, if we can bound the diameter of the polytope, then we can reduce many problems to a version that can be analyzed with the help of a computer.

For example, for a fixed D, there are only a finite number of lattice polygons with diameter less than D. Thus, the computer can generate all lattice polygons with diameter less than D. These can then be analyzed to get results concerning other parameters, such as the width or the area. Such results have then been established with the assistance of the computer.

As an example, consider the following simple result: Any lattice polygon with 4 interior lattice points has perimeter not smaller than $4\sqrt{5}$. To prove a result such as this, we must first give a constructive mathematical proof that there exists some constant D such that all lattice polygons with perimeter at most $4\sqrt{5}$ have diameter at most D. Then we need only find all polygons with diameter at most D and verify that the perimeter is not smaller than $4\sqrt{5}$.

In order to find all lattice polygons with diameter at most D, it is sufficient to devise a computer algorithm for generating all such convex lattice polygons. Furthermore, to be practical, the algorithm must be efficient. We describe below such an algorithm.

Appendix B shows the FORTRAN code that implements this algorithm. The algorithm embodies several techniques that enable it to run reasonably fast. (On a VAX-11/780 it was able to find all lattice polygons with $D \leq 10$ and $g \leq 10$ in under 2 hours of CPU time). Since such polygons typically have at least 8 vertices, 7 of which can occur at any of 100 lattice points, a brute-force rough estimate would show that at least 100^7 or 100 trillion polygons had to be examined. Even if each polygon could be examined in 10 microseconds, it would still take over 30 years of CPU time to examine them all. Thus, it is necessary to prune the search and find a quick way to eliminate those polygons that are not convex or have area 0, etc.

The general method of approach is to use what's known as a *backtrack algorithm*. For general information about backtrack algorithms, consult Nijenhuis and Wilf [73], chapter 27. In the backtrack algorithm, we start by placing the first vertex of the polygon at the origin. We then lay the next vertex down at (1,0). Then we generate all possible polygons with these two points as vertices. When that is all done, we back up (or backtrack) to the vertex we placed at (1,0). We remove it from there and place it at the next possible position, (2,0).

We continue in this manner at each stage. At every step, we already have, say, n vertices placed in the plane. We then find all possible polygons that continue this one. When that is done, we back up to the nth vertex and move it to its next spot. When there are no more places for the nth vertex, we remove it completely and back up to the $(n-1)$th vertex which gets moved to its next position, etc.

For coding a backtrack algorithm, it makes more sense to use a finite state automaton as the model rather than a traditional structured programming approach. In this model, the program can be in various states, such as the state of moving a vertex to a new position, or the state of backing up to a previous vertex. After each new state is entered, the program performs the necessary actions associated with that state and then moves into a new state. The concept of moving to a new state is modelled in the program by branching to the appropriate piece of code that handles the new state.

In this particular program (LATTICE), there are three main states. State 1 (PLACE NEW VERTEX) is "go place the next vertex" (line 20 in the code). State 2 (MOVE VERTEX) is "move the current vertex to its next possible location" (line 30 in the code). State 3 (BACKTRACK) is "back up to the previous vertex" (line 40 in the code). There is also an initialization state and a "done" state. The finite state machine begins in the INITIALIZATION state.

Algorithm LATTICE. Positive integers g and D are specified as input parameters. We are interested in generating all convex lattice polygons with g or fewer interior lattice points, and diameter not larger than D.

Two lattice polygons that are lattice congruent are to be considered the same and only one of them need be generated. Since we are interested in some properties (such as width and perimeter) that are not invariant under integral unimodular affine transformations, we cannot consider lattice equivalent polygons to be the same and must generate them all. There is no harm in generating some polygons more than once, as long as this does not occur often enough to slow down the program significantly.

Since we can translate the lattice polygon so that one vertex lies at the origin, it suffices to pick the initial vertex (vertex 0) to be at (0,0) and to never move it. Furthermore, we can rotate the polygon so that it lies completely on or above the x-axis. We know from Proposition 2.5 that it suffices to look at polygons contained within the first quadrant only, however, optimizations to be described later made it easier to allow the polygon to lie in quadrants 1 and 2. Furthermore, Proposition 2.5 talks about lattice equivalence, not lattice congruence, so we cannot generate all possible lattice polygons in quadrant one only and still keep the constraint that one vertex lie at the origin. Since we are only interested in polygons with diameter not greater than D, it suffices to restrict all vertices to have ordinate not more than D and abscissa between $-D$ and D inclusive. Thus, the polygon lies within a rectangle of width $2D$ and height D. We call this rectangle the *target rectangle*.

Section 4.1.
INITIALIZATION state.

Since the program will have to compute the number of lattice points on the boundary of the polygon, a large number of GCDs will have to be computed (using Proposition 1.2.1). To make this step efficient, we can pre-compute all the GCDs needed. Since the difference between the x or y coordinates of any pair of vertices is between $-2D$ and $+2D$, there are only a finite number of GCDs that might ever have to be computed. We compute these all ahead of time using the Euclidean algorithm and the relation $\gcd(m, n) = \gcd(m - n, n)$ and store all these precomputed GCDs in a table. We can then access this table any time we need to figure out the GCD of two numbers and this will be much faster than going through the Euclidean algorithm each time.

Instead of calculating the area of each polygon obtained, which would be slow, we incrementally calculate the area as we go along. At each vertex, v_n, we store the quantity AREA[n], which represents the area of the polygon generated thus far. Whenever we add a new vertex to the polygon, we need only calculate the area of the triangle formed by the new edge (and the line back to the origin) and add this to the previously stored area. When backing up to a previous vertex, no additional calculation has to be performed since we have already stored the area to that point at the vertex. (Actually, since the area is always a multiple of $1/2$, we store twice the area, so that we can use integer arithmetic throughout.)

In a similar manner, we associate the quantity, BOUNDARY[n], at vertex v_n. This variable holds the number of lattice points found on the boundary of the polygon thus far.

The initialization code sets the vertex number, VERTEX, to 0 and places this vertex at the origin. The AREA and BOUNDARY values associated with vertex 0 are set to 0.

Section 4.2.
PLACE NEW VERTEX state.

The VERTEX number is incremented by one. This new vertex is initially placed at the bottom left hand corner of the target rectangle. However, as an optimization, vertex 1 starts off at (0,1).

We need to prove that we do not miss any polygons by this procedure.

Proof. Suppose some convex lattice polygon fits inside our target rectangle but has a vertex on the positive x-axis (as well as at the origin). This makes the bottom edge of the polygon lie on the x-axis. Since the diameter of the polygon is not more than D, the length of this edge is not more than D. Thus we can translate this polygon to the left until the right endpoint of this edge lies at the origin. The left endpoint must then also lie within the target rectangle. $\square$

It is because of this optimization (and the constraint that one vertex lies at the origin) that we allow the vertices to lie in a $2D$ by D rectangle instead of in a D by D square centered about the y-axis.

Since from this state we will make a transition to the MOVE THE VERTEX state and immediately move the vertex, we actually set the initial vertex location to be one less than where we really want it initially. In that way, the next state will begin by moving it to its desired initial location.

Section 4.3.

MOVE THE VERTEX state.

The current vertex gets moved to its next location. In general, a vertex starts at the lower left corner of the target rectangle and then keeps moving right until it hits the right boundary. At that point, it moves up one row and over to the lefmost column. This continues until the vertex reaches the rightmost position of the topmost row.

So this state starts out by incrementing the x-coordinate of the current vertex by 1. However, if we are already at the right edge of the target rectangle, then we bump the y-coordinate by 1 and set the x-coordinate back to the leftmost edge of the target rectangle.

As a further optimization, we note that if the polygon is sloping downward; i.e., YCOORD[VERTEX-1] $\leq$ YCOORD[VERTEX-2], and VERTEX>1, then we may as well abort if we try to place the next vertex at y-coordinate YCOORD[VERTEX-1] or higher. Such a placement is guaranteed to make the polygon non-convex. If such happens or if we move above the topmost row of the target rectangle, we make a transition to the BACKTRACK state.

Now we make various checks to see if the proposed location is a valid place for this vertex. Three conditions are checked.

Condition 1 checks to see if we are trying to move to the origin. Since we have proven that we need not consider any polygons with a vertex on the positive x-axis, and since there is already a vertex at the origin, if we are about to move a vertex to the origin, we move it up one row instead and over to the leftmost column of the target rectangle.

Condition 2 checks to make sure that the new vertex is not collinear with the previous vertex and the origin and that these 3 vertices are properly oriented counterclockwise. If the coordinates of the nth vertex are (x_n, y_n), then this condition can be expressed by the equation

$$A = \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} > 0$$

for $n > 1$, where A is the area of the new triangle added to the polygon by the addition of this vertex. If this condition fails, then we try to move the vertex again by reentering the MOVE VERTEX state.

Condition 3 checks to make sure that the new vertex is not collinear with the previous two vertices and that the polygon is still sloping in the proper direction. We insist that the vertices be oriented counterclockwise as you move from one vertex to successively higher-numbered vertices. If the coordinates of the nth vertex are (x_n, y_n), then this condition can be expressed by the equation

$$\begin{vmatrix} x_{n-2} & y_{n-2} & 1 \\ x_{n-1} & y_{n-1} & 1 \\ x_n & y_n & 1 \end{vmatrix} > 0$$

for $n > 1$. For $n = 1$, the condition is $x_n \geq 0$. If this condition fails, then we try to move the vertex again by reentering the MOVE VERTEX state.

As another optimization, we handle vertex 1 specially. If its x-coordinate is ever negative then we force it to be 0. That is, we avoid half of all possible positions for vertex 1 in the target rectangle. We must prove that we do not miss any polygons by this optimization.

Proof. Suppose we had a polygon with vertex 0 at the origin and vertex 1 strictly inside quadrant 2. Then, since the vertices of the polygon are oriented in a counter-clockwise manner, this implies that the entire polygon lies in quadrant 2. We may thus apply a reflection about the y-axis to get another polygon located in quadrant 1 which is lattice congruent to the original polygon. This polygon will be found by the search and so there is no loss in discarding polygons lying entirely in quadrant 2. □

Now that we have a valid location for this new vertex (v_n), we update AREA[n] and BOUNDARY[n]. We can then use Pick's Formula to update INTERIOR, the number of lattice points inside the polygon. If this number is too large (greater than g), we make a transition to the BACKTRACK state.

A few more optimizations can now be performed. We ignore any polygons whose first edge makes a larger angle with the y-axis than with the newly formed edge. (If such is the case, we make a transition to the PLACE NEW VERTEX state.) We must prove that this does not cause us to skip any polygons.

Proof. Note that if this condition occurs, we move to the PLACE NEW VERTEX state not to the MOVE VERTEX state. In other words, we are only rejecting polygons where the first edge makes a larger angle with the positive y-axis than the last edge of that polygon makes with the positive y-axis. For given any such polygon, we can reflect it about the y-axis and get another polygon, lattice congruent to the original one, in which the first edge forms an angle with the positive y-axis that is smaller than or equal to the angle that the last edge makes with the positive y-axis. □

This angle condition is easily tested for without the necessity of calculating arc tangents since all vertices are at lattice points and in the range 0 to $\pi/2$, the tangent function is monotone. The condition can be expressed algebraically as $y_n > 0$ AND $x_n \leq 0$ AND $x_1 y_n > -x_n y_1$. No floating point calculations are needed.

Once we have a valid polygon, we compute various attributes of the polygon, such as the area and diameter and write these out to a file, together with the coordinates of the vertices of the polygon. A report-writing program will later scan this file and create a report based on its findings.

Two of the parameters calculated, namely the horizontal and vertical widths, enable us to make another optimization. If the vertical width is larger than the horizontal width, then the polygon is rejected and we make a transition to the MOVE VERTEX state. We must show that this does not cause us to lose any polygons.

Proof. If we have a polygon whose vertical width is larger than its horizontal width, we can rotate the polygon through $\pi/2$ radians. Then translate it so that the lowest vertex lies at the origin. This resulting polygon has a vertical width less than or equal to the horizontal width, is lattice congruent to the original polygon, and will be found by the algorithm. □

After the data has been written out, we make a transition to the PLACE NEW VERTEX state.

Section 4.4.
BACKTRACK state.

VERTEX is decremented by 1 to cause us to back up to the previous vertex. If VERTEX is now 0, this means we are done and we make a transition to the DONE state. Otherwise, we make a transition to the MOVE VERTEX state.

Section 4.5.
DONE state.

The output file is closed and the program cleans up and exits.

Section 4.6.
Other optimizations.

A boolean variable called INSIDE-FLAG is maintained to allow us to make additional optimizations. This variable is set to TRUE any time the current vertex is placed at a valid spot.

This variable is set to FALSE in the initialization state. It is set to TRUE in the BACKTRACK state. It is set to FALSE in the PLACE NEW VERTEX state and it is set to FALSE anytime the MOVE VERTEX state moves a vertex up to a new row.

Anytime that one of conditions 1, 2, or 3 applies and we go back to the MOVE VERTEX state, we first check the INSIDE-FLAG. If INSIDE-FLAG is TRUE, then we bump the XCOORD to its largest value, so that when we get to the MOVE VERTEX state, it will start by moving the vertex up one row (and to the left edge of the target rectangle). We can do this because we know that if the polygon is rejected because of one of these 3 conditions, the same conditions will still apply if we merely move the last vertex to the right, and hence the subsequent positions of that vertex on the current row yield polygons that will be rejected and thus can be skipped.

Section 4.7.
Report Printer.

A report printer was also written (the code is not shown here). It reads the data produced by the LATTICE algorithm and locates various maxima and minima, such as polygons with a given g and minimum diameter. It also calculates other parameters such as the (minimal) width.

Section 5
Inequalities involving interior lattice points

In this section, we will be discussing inequalities for convex lattice polygons containing g interior lattice points.

Section 5.1.
Inequalities for boundary lattice points.

Proposition 5.1.1 (minimum b). *Let K be a convex lattice polygon in the plane with g interior lattice points and b lattice points on its boundary. Then $b \geq 3$. Furthermore, we can always find a lattice triangle with $b = 3$ that contains exactly g interior lattice points.*

Proof. Trivially, $b \geq 3$ since, by definition, a polygon must have at least 3 vertices and these must be lattice points. We have already shown in section 3 (Proposition 3.4) that for any non-negative integer g, we can find a lean lattice triangle with exactly g interior lattice points. However, it is instructive to give another proof.

Let $X = (0, 0)$, $Y = (1, 0)$, and $Z = (h + 2, h)$ where $h = 2g + 1$. We will show that this triangle contains exactly g interior lattice points.

We note that h must be odd since g is an integer, so $\gcd(h, h+2) = \gcd(h, 2) = 1$. Thus segment XZ contains no interior lattice points. Clearly, $\gcd(h + 1, h) = 1$, so segment YZ has no interior lattice points. Segment XY is of length 1 and so has no interior lattice points. Thus $b = 3$. The area of triangle XYZ is one half the base times the altitude $(h/2)$. But by Pick's Formula, the number of lattice points inside the triangle is

$$A - \frac{b}{2} + 1 = \frac{h}{2} - \frac{3}{2} + 1 = \frac{2g+1}{2} - \frac{1}{2} = g,$$

so triangle XYZ has precisely g interior lattice points, as desired. $\square$

In 1976, Scott found the upper bound for b, for any given $g > 0$. We present his result below.

Theorem (Scott's Bound for b). *Let K be a convex lattice polygon in the plane with $g > 0$ interior lattice points and b lattice points on its boundary. Then $b \leq 2g+7$. Equality holds when and only when K is lattice equivalent to TRIANG$(3, 3)$.*

The following is essentially Scott's proof. For later use, we repeat his proof with a slightly more detailed analysis of the equality conditions.

Proof. (Scott [100])

Let

$$f(g) = b - 2g.$$

With g fixed, an upper bound for b will be found if we find a lower bound for f.

Applying Pick's Formula, $A = b/2 + g - 1$ leads us to two alternate expressions for f:

$$f(g) = 2b - 2A - 2$$

and

$$f(g) = 2A - 2g + 1.$$

Let the two horizontal support lines for K be $y = 0$ and $y = h$, $h > 0$. These two support lines meet K in segments of lengths p and q (possibly 0). Since K contains interior lattice points, we must have $h \geq 2$.

The two "sides" of K each travel from $y = 0$ to $y = h$ and have no horizontal portions, since K is convex. So each side can have at most one lattice point for each horizontal lattice line between $y = 0$ and $y = h$. We deduce that

$$b \leq p + q + 2h.$$

Equality holds if and only if both sides meet all parallels at lattice points.

Since K is convex, it contains the convex hull of the two horizontal segments. Thus

$$A \geq \frac{h(p+q)}{2}.$$

Equality holds if and only if both sides are straight lines. So

$$f(g) = 2b - 2A - 2 \leq 2(p + q + 2h) - h(p + q) - 2 = (p + q - 4)(2 - h) + 6.$$

Equality holds if and only if both sides meet all parallels at lattice points and both sides are straight lines.

We wish to show that $f(g) \leq 7$. We now consider 5 cases.

Case 1. $h = 2$.

Then $f(g) \leq (p+q-4)(2-h)+6 = 6$. Equality holds if and only if both sides meet all parallels at lattice points and both sides are straight lines.

Case 2. $h \neq 2$, $p + q \geq 4$.

Then since $2 - h \leq 0$, $f(g) \leq (p + q - 4)(2 - h) + 6 \leq 6$. Equality holds if and only if both sides meet all parallels at lattice points and both sides are straight lines and $p + q = 4$.

Case 3. $h = p + q = 3$.

Then $f(g) \leq (p + q - 4)(2 - h) + 6 = 1 + 6 = 7$. Equality holds if and only if both sides meet all parallels at lattice points and both sides are straight lines.

Case 4. $h = 3$ and $p + q \leq 2$.

In this case, $b \leq p + q + 2h \leq 8$. But since $g \geq 1$, $f(g) = b - 2g \leq 8 - 2 = 6$. Equality holds if and only if both sides meet all parallels at lattice points and $p + q = 2$ and $g = 1$.

Case 5. $h \geq 4$ and $p + q \leq 3$.

Let P be any point where K meets the x-axis and let Q be any point where K meets the support line $y = h$. Let vertical support lines $x = 0$ and $x = h'$ meet K at points R and S, respectively. We may assume that $h' \geq h$, for if not, we can rotate the figure through $\pi/2$ and relabel the parts so that this is true. If after the relabeling, we find that either $h < 4$ or $p + q > 3$, we are done because the figure falls into one of the previous cases.

By Lemma 2.6, we can find a shear (leaving the x-axis fixed) which transforms the figure into one in which the abscissae of P and Q differ by u with

$$0 \leq u \leq \frac{h - p - q}{2}.$$

This shear leaves A, b, h, and $p + q$ unchanged, and preserves the convexity of K. Again, if h' winds up less than h, rotate the figure by $\pi/2$, interchanging the roles of h and h'. (If h then gets to be smaller than 4, we are done by one of the earlier cases.)

Since this resulting figure is convex, we have

$$A \geq A(PQRS) = \frac{1}{2}PQ(r_1 + r_2)$$

with equality if and only if the figure is a quadrilateral (which implies $p = q = 0$). Since $PQ \geq h$, $r_1 \geq TU$, and $r_2 \geq PV$ (with equality if and only if PQ is vertical), we get

$$A \geq \frac{1}{2}h(h' - u).$$

Equality holds if and only if the figure is a quadrilateral and PQ is vertical. Since $h' \geq h$, we find

$$A \geq \frac{1}{2}h(h - u).$$

Applying the upper bound for u, we see that

$$A \geq \frac{1}{4}h(h + p + q).$$

Equality holds if and only if the figure is a quadrilateral and PQ is vertical. Hence

$$f(g) = 2b - 2A - 2 \leq 2(p+q+2h) - \frac{1}{2}h(h+p+q) - 2 = \frac{1}{2}(p+q)(4-h) + \frac{1}{2}h(8-h) - 2 \leq 6$$

since $h \geq 4$ and $h(8-h)$ assumes its maximum value of 8 when $h = 4$. Equality here requires that the two "sides", PRQ and PSQ are straight lines, which cannot hold at the same time as the other restrictions. Thus equality is not possible in this case.

Thus, in all cases, we have $f(g) \leq 7$. Equality only holds for case 3, in which case $h = 3$, $p + q = 3$, $b = 9$, $A = 9/2$ and both sides are straight lines meeting all the parallels at lattice points. In other words, the figure is a trapezoid of height h and bases of lengths p and q, with $p+q = 3$. Since we may as well assume $p > q$, the only cases are $p = 3$, $q = 0$ or $p = 2$, $q = 1$. The first case yields a triangle equivalent to TRIANG$(3, 3)$. The second possibility is easily seen to be impossible, for there are only 3 inequivalent possibilities for the placement of the segment of length q, and none of them yield both sides meeting all the parallels at lattice points. $\square$

Proposition 5.1.2 (maximum b). *Let K be a convex lattice polygon in the plane with g interior lattice points and b lattice points on its boundary.*

a. *If $g = 0$, then b can be an arbitrary positive integer satisfying $b \geq 3$.*
b. *If $g = 1$, then $b \leq 9$. Equality holds when and only when K is lattice equivalent to TRIANG$(3, 3)$.*
c. *If $g > 1$, then $b \leq 2g + 6$ and this inequality is best possible.*

Proof.

a. If $b = 2k$, then consider the lattice rectangle, TRAP$(k, k, 1)$, whose vertices are at $(0, 0)$, $(0, 1)$, $(k - 1, 1)$, and $(k - 1, 0)$. Since $b \geq 3$, we must have $k \geq 1$. This lattice rectangle then has exactly $b = 2k$ lattice points on the boundary and no interior lattice points. If $b = 2k+1$, then consider the lattice trapezoid, TRAP$(k, k - 1, 1)$ whose vertices are at $(0, 0)$, $(0, 1)$, $(k, 0)$, and $(k - 1, 1)$. Again, $k \geq 1$ and this trapezoid has the desired properties.
b. This is equivalent to Scott's bound for b.
c. By Scott's bound for b, $b \leq 2g + 7$. If equality were to hold, then K would be equivalent to TRAP$(3, 3)$ and we would have $g = 1$. But since $g > 1$, equality can't hold, and so we must have $b \leq 2g + 6$. Furthermore, b can equal $2g + 6$ as can be seen by TRIANG$(4, 2)$. $\square$

Proposition 5.1.3 (range of b). *Let K be a convex lattice polygon in the plane with g interior lattice points and b lattice points on the boundary. Then b may be any positive integer within (and including) the bounds specified by the preceding two propositions.*

Proof. Consider the lattice pentagon, K, with vertices $P_1 = (0,0)$, $P_2 = (p,0)$, $Q_1 = (0,2)$, $Q_2 = (q,2)$, and $R = (g+1,1)$ where p and q are positive integers satisfying $0 \leq p \leq g+1$ and $0 \leq q \leq g+1$. Then P_1Q_1 contains 3 lattice points, P_1P_2 contains an additional p lattice points, Q_1Q_2 contains an additional q lattice points, and P_2R and Q_2R contain no other lattice points besides R. Thus $b(K) = p+q+4$. But $p+q$ ranges from 0 to $2g+2$ inclusive, so b ranges from 4 to $2g+6$ inclusive. Hence if b is any value from 4 to the maximum possible value for b of $2g+6$, we can find p and q so that K contains b lattice points on the boundary. Note finally that K contains exactly g lattice points in the interior.

The only other possible value for b is $b = 3$ and we have already constructed a minimal lattice polygon with $b = 3$ and g interior points. (See Proposition 3.4.) □

Scott did not bother to characterize all lattice polygons for which equality held in the formula $b \leq 2g+6$. We do so now by a careful examination of his proof.

Theorem 5.1.4 (extremal figures). *If $g > 1$ and $b = 2g+6$ then K is lattice equivalent to one of the following polygons:*

a. *TRIANG$(4,4)$.*
b. *TRIANG$(2g+2,2)$.*
c. *TRAP$(p,q,2)$ with $p+q = 2g+2$.*

Proof. We go back to the proof of Scott's bound for b and in each of the five cases, check when equality can hold for $b = 2g+6$.

Case 1. $h = 2$.

Here we had $f(g) \leq 6$ with equality if and only if both sides were straight lines that met all parallels at lattice points. Thus K is a trapezoid with bases p and q and height h and $b = p+q+2h = p+q+4$. By Pick's Formula, $A = b/2+g-1 = (p+q)/2+g+1$ and from the formula for the area of a trapezoid, $A = h(p+q)/2 = p+q$. Equating these shows that $p+q = 2g+2$. We can translate the trapezoid so that the base starts at the origin and extends to the right, and the smaller base lies along $y = 2$. Since the height is 2, there are really only 2 inequivalent positions for the leftmost vertex of the top base: $(0,2)$ and $(1,2)$. Point $(1,2)$ is ruled out because then the left side does not meet the $y = 1$ parallel at a lattice point. Thus the figure is equivalent to TRAP$(p,q,2)$ with $p+q = 2g+2$, or TRIANG$(2g+2,2)$ in the case where $q = 0$.

Case 2. $h > 2$, $p+q \geq 4$.

In this case we also had $f(g) \leq 6$ with equality if and only if $p+q = 4$ and both sides are straight lines meeting all the parallels at lattice points. This makes the figure a trapezoid. Translate the leftmost vertex of the base to the origin and the rightmost base to $A = (p,0)$. Since the left side meets each parallel at a lattice point, let B be the point where it meets $y = 1$. A suitable shear yields an equivalent trapezoid wth B at $(0,1)$ and O and A unchanged. Let the trapezoid be $OADC$ and we may as well assume that $OA \geq CD$ (otherwise perform a reflection about a horizontal line that interchanges the bases). The only way for AD to meet all parallels at lattice points is if it extends straight up vertically (in which case $OA = CD = 2$) and a rotation through $\pi/2$ shows that this figure is equivalent to TRAP$(g+1,g+1,2)$, a case previously covered; or if AD has a slope of -1 (a slope of -1/2 or less would cause AD to cross the y-axis at a height of less than 3). For a slope of -1, we see that the resulting figure is either TRIANG$(3,3)$ or TRIANG$(4,4)$.

Case 3. $h = p + q = 3$.

In this case, $f(g) = 7$ so $f(g)$ can not equal 6.

Case 4. $h = 3$ and $p + q \leq 2$.

In this case, $f(g) \leq 6$ and equality holds if and only if $g = 1$ and both sides meet all parallels at lattice points. Translate the figure so that one base has its leftmost endpoint at the origin and the rightmost endpoint is at $A = (p, 0)$. Both sides meet the parallel $y = 1$ at lattice points, say B and C. If B and C are consecutive lattice points, then there is no way the polygon can contain an interior lattice point. If B and C are separated by more than one lattice point, we will have $g > 1$, a contradiction. Thus the unique interior lattice point, E, lies on $y = 1$ between B and C. Perform a shear leaving the x-axis fixed and taking E into $(1, 1)$. We may as well assume that $p \geq q$, so that there are two cases, $p = q = 1$ and $p = 2$, $q = 0$. In the first case, A is at $(1, 0)$ and B and C are at $(0, 1)$ and $(2, 1)$ respectively. It is easy to see that there is no way the polygon can be extended further upward to a height of 3 and still have the sides (not necessarily straight) meet the parallel $y = 2$ at consecutive lattice points. In the second case, $A = (2, 0)$, $B = (0, 1)$ and $C = (2, 1)$. We may extend the polygon upward by drawing straight lines to $D = (0, 3)$ from B and C. This is the only way to meet the parallel $y = 2$ at consecutive lattice points and reach a height of 3. This resulting figure is equivalent to TRAP$(3, 1, 2)$ and has already been enumerated since $3 + 1 = 4 = 2g + 2$.

Case 5. $h \geq 4$ and $p + q \leq 3$.

In this case we saw that $f(g)$ was strictly less than 6 since equality could not hold or that the figure was equivalent to one of the previous cases. □

Proposition 5.1.5. *If $g > 1$, then $b \leq A + 4$.*

Proof. If $g > 1$, then we already know that $b \leq 2g + 6$. But from Pick's Formula, we have $g = A - b/2 + 1$. Thus $b \leq 2A - b + 2 + 6$ or $b \leq A + 4$. □

We mention now a few miscellaneous results.

Result 5.1.6. *If every angle of a lattice polygon is obtuse, then $b \leq 2g$, and this result is best possible.*

Reference. [Scott, 1979].

Result 5.1.7. *If every angle of a lattice polygon is acute, then $b \leq 2g + 4$, and this result is best possible.*

Reference. [Scott, 1979].

Result 5.1.8. *If K is centrally symmetric and $g = 1$, then $b \leq 8$.*

Reference. [Wills, 1981].

Very little is known about analogues of Scott's Bound for b in E^n for $n > 2$. We mention a few known results.

Result 5.1.9. *If K is a centrally symmetric convex body in E^n and $g(K) = 1$, then $V(K) \leq 2^n$ and $G(K) \leq 3^n$. If $g(K) = 1$ and $V(K) = 2^n$, then K is a convex polytope with at most $2^{n+1} - 2$ facets.*

Result 5.1.10. *Let $g(n,g)$ and $v(n,g)$ be defined for $g \geq 1$ and $n \geq 2$ as follows: $g(n,g) = \sup\{G(P), P \subset E^n, g(P) = g\}$ and $v(n,g) = \sup\{V(P), P \subset E^n, g(P) = n\}$ where the sup is taken over all lattice polytopes in E^n. Then*

a. *$g(2,1) = 10$ and $g(2,g) = 3g + 6$ for $g \geq 2$.*
b. *For all n, $n \geq 4$, and all g, $g \geq 1$,*

$$v(n,g) \geq \frac{g+1}{n!} 2^{2^{n-a}}$$

and

$$g(n,g) \geq \frac{g+1}{6(n-2)!} 2^{2^{n-a}},$$

where $a = 0.5856....$

c. *$v(3,g) \geq 6(g+1)$ and $g(3,g) \geq 16g + 23$.*
d. *$v(4,1) \geq 147$ and $g(4,1) \geq 680$.*

References.

a. This follows from Scott's Bound for b.
b-d. Zaks, Perles, and Wills [128].

Conjecture (Zaks, Perles, and Wills). *$v(n,g) < \infty$ for all $n \geq 3$ and $g \geq 1$.*

This conjecture implies that $g(n,g) < \infty$ because of the following result due to Blichtfeldt ([68], p. 55).

Result 5.1.11. *If P is a lattice polytope in E^n, then $G(P) \leq n!V(P) + n$.*

Section 5.2.
Inequalities for Area.

Proposition 5.2.1 (minimum A). *Let K be a convex lattice polygon in the plane with g interior lattice points and area A. Then $A \geq g + 1/2$. Furthermore, we can always find a lattice polygon with area $A = g + 1/2$ that contains exactly g interior lattice points. Equality holds when and only when K is a lean triangle.*

Proof. By Pick's Formula, we have $A = b/2 + g - 1$. By Proposition 5.1.1, we have $b \geq 3$, so $A \geq g + 1/2$. We can also find a lattice triangle with exactly g interior points and $b = 3$ by Proposition 3.4, so for this lattice triangle we have the equality $A = g + 1/2$. □

Proposition 5.2.2 (maximum A). *Let K be a convex lattice polygon in the plane with g interior lattice points and area A.*

a. *If $g = 0$, then A can be an arbitrary positive integral multiple of $1/2$.*
b. *If $g = 1$, then $A \leq 9/2$. Equality holds when and only when K is lattice equivalent to TRIANG$(3, 3)$.*
c. *If $g > 1$, then $A \leq 2g + 2$.*

Proof.

a. The lattice triangle with vertices $(0, 0)$, $(0, 1)$, and $(A, 0)$ where A is an arbitrary positive integer has area $A/2$ and contains no interior lattice points.
b. By Scott's bound for b, we have $b \leq 9$ when $g = 1$, so by Pick's Formula, $A = b/2 + g - 1 \leq 9/2$, with equality as before.
c. Again, combining Scott's bound for b with Pick's Formula gives $A = b/2 + g - 1 \leq (2g + 6)/2 + g - 1 = 2g + 2$, with equality as before. □

Proposition 5.2.3 (range of A). *Let K be a convex lattice polygon in the plane with g interior lattice points and area A. Then the area of K may be any integral multiple of $1/2$ within (and including) the bounds specified by the preceding two propositions.*

Proof. Given a value of A in the allowable range, we can use Pick's Formula to calculate the value of b. This b will be in the allowable range for b, so by Proposition 5.1.3, we can find a convex lattice polygon, K, with this value for b and exactly g interior lattice points. This polygon has the proper value for A. □

Proposition 5.2.4 (extremal figures). *Let g be an integer with $g > 1$. Then we know that $A \leq 2g + 2$. Equality holds when and only when K is lattice equivalent to TRIANG$(4, 4)$ or to TRIANG$(2g + 2, 2)$ or to the trapezoid, TRAP$(p, q, 2)$, with $p + q = 2g + 2$.*

Proof. If $A = 2g + 2$, then by Pick's Formula, we have $b/2 + g - 1 = 2g + 2$ or $b = 2g + 6$ (and conversely). But for $g > 1$, we have shown in Proposition 5.1.4 that $b = 2g + 6$ if and only if K is one of the polygons described by proposition 5.2.4. □

If $g = 1$, then the above-mentioned figures are also minimal as well as one additional triangle, namely TRIANG$(3, 3)$.

Section 5.3.
Inequalities for inner boundary points.

We can divide the lattice points on the boundary of a lattice polygon into two sets. Those that are in the relative interior of an edge of the polygon we call *inner boundary points*. Those that are at the end points of an edge of the polygon we call *vertices*.

We denote the number of inner boundary points by b_0 and the number of vertices by v. Thus $b = b_0 + v$.

Proposition 5.3.1 (minimum b_0). *Let K be a convex lattice polygon in the plane with g interior lattice points and b_0 inner boundary points. Then $b_0 \geq 0$ and this bound is best possible (for we can always find a lattice polygon that contains exactly g interior lattice points and has no inner boundary points).*

Proof. The proof is the same as the proof of the proposition about the minimum value of b. In that proof, we exhibited a lattice triangle with no inner boundary points and exactly g interior lattice points, namely WEDGE(g). $\square$

Proposition 5.3.2 (maximum b_0). *Let K be a convex lattice polygon in the plane with g interior lattice points and b_0 inner boundary points.*

a. *If $g = 0$, then b_0 can be arbitrarily large.*
b. *If $g = 1$, then $b_0 \leq 6$ and this bound is best possible.*
c. *If $g > 1$, then $b_0 \leq 2g + 3$ and this bound is best possible.*

Proof. This proposition is an immediate consequece of Scott's bound for b since every polygon must have at least 3 vertices. Alternatively, we can note that TRIANG$(2g + 2, 2)$ contains exactly g interior lattice points and has $2g + 1$ inner boundary points. $\square$

Section 5.4.
Inequalities for the longest edge.

Let s denote the number of lattice points in the relative interior of the side of the polygon that contains the most number of lattice points.

First, we note an obvious fact.

Observation. *If n is a positive integer, then a horizontal line of length greater than $n + 1$ must contain $n + 1$ points in the relative interior with integer x-coordinates.*

Proposition 5.4.1 (Maximum s). *If a convex lattice polygon has exactly g interior lattice points, $g > 0$, then no side of the polygon contains more than $2g + 1$ lattice points interior to it.*

In other words, $g > 0 \Rightarrow s \leq 2g + 1$.

Proof. Let K be the lattice polygon and let $S = S_1S_2$ be a particular side of K. Suppose that S contains s lattice points in its interior, with $s > 2g + 1$. We will show that we reach a contradiction.

We can find an integral unimodular affine transformation that takes S into the x-axis line with no portion of K below it. Since this preserves the number of lattice points in K and S, we may assume without loss of generality that K is already placed in this manner.

Since $g > 0$, there must be some point $S_0 \in K$ whose height, h, above the x-axis is more than 1. Let the line $y = 1$ meet S_0S_1 at point P and S_0S_2 at Q. From similar triangles S_0PQ and $S_0S_1S_2$ we have

$$\frac{h}{s+1} = \frac{h-1}{PQ}$$

or

$$PQ = \frac{h-1}{h}(s+1) \geq (s+1)/2 > (2g+2)/2 = g+1.$$

Thus segment PQ has length strictly greater than $g+1$, so by the observation, it must contain $g+1$ interior lattice points. Thus we have found more than g lattice points interior to K, a contradiction. $\square$

Notation. For g a positive integer, we let $s(g) = \max\{g(E)|E$ is an edge of a convex lattice polygon K and $g(K) = g\}$.

Proposition 5.4.2 (Extremal figures). *With $s(g)$ as defined above (and $g > 0$), $s(g) \leq 2g+1$ and equality holds when and only when the polygon is lattice equivalent to TRIANG$(2g+2, 2)$. If $g = 0$, then $g(E)$ is unbounded.*

Proof. The previous proposition shows that $s(g) \leq 2g+1$. In the notation of the previous proof, if $s = 2g+1$, then $PQ \geq g+1$ with equality if and only if $h = 2$.

If $g = 0$, we can make the sides arbitrarily large. Just consider the lattice triangle with vertices $(0,0)$, $(g,0)$, and $(1,0)$. This triangle has no interior lattice points and the number of lattice points on the base can be any non-negative integer. $\square$

Proposition 5.4.3 (Minimal s). $s \geq 0$. *Equality is possible for all g.*

Proof. We have already shown how to find a lattice triangle with $b = 3$ for any g (Proposition 3.4). For such a triangle, $s = 0$. $\square$

Section 5.5.
Inequalities for Perimeter.

Let P denote the perimeter of the lattice polygon.

Proposition 5.5.1 (P unbounded). *For any positive integer, g, there are lattice polygons with g interior lattice points and arbitrarily large perimeter.*

Proof. Consider the lattice triangle OAB where $O = (0,0)$, $A = (2g+2,0)$, and $B = (0,2)$. Applying a shear of magnitude k, leaving the x-axis fixed, we find that O and A remain fixed, and B moves to $(2k,2)$. This triangle has g interior lattice points and has a perimeter larger than $OB = \sqrt{4k^2+4}$. But we can make $\sqrt{4k^2+4}$ arbitrarily large by making k arbitrarily large. Thus, P is unbounded. $\square$

We investigated by computer the relationship between P and g for all convex lattice polygons with $D \leq 10$ and $g \leq 10$. In each case, we found the minimum value of P, which was always less than 14.

Proposition 5.5.2 (effectiveness of search for minimum P). *No lattice polygons with $P \leq 14$ were missed by the computer search.*

Proof. Consider any polygon with $g \leq 10$ and $P \leq 14$. Then using the fact that $D \leq P/2$ (Proposition 1.3.1d) for all convex bodies, we would have $D \leq 7$. Thus the polygon would have been found in our search since we searched all polygons with $D \leq 10$. $\square$

The following results were found by computer:

Proposition 5.5.3. *If $g = 0$ then $P \geq 2+\sqrt{2} \approx 3.414$. Equality occurs when and only when K is lattice congruent to the isosceles right triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$. See figure 5.5-1.*

```
o .
o o
```

```
Figure 5.5-1
Unique polygon with g=0
and smallest perimeter
```

Proposition 5.5.4. *If $g = 1$ then $P \geq 4\sqrt{2} \approx 5.657$. Equality occurs when and only when K is lattice congruent to the diamond with vertices $(0,1)$, $(1,0)$, $(1,2)$, $(2,1)$. See figure 5.5-2.*

```
. o .
o . o
. o .
```

```
Figure 5.5-2
Unique polygon with g=1
and smallest perimeter
```

Proposition 5.5.5. *If $g = 2$ then $P \geq 2\sqrt{2}+2\sqrt{5} \approx 7.301$. Equality occurs when and only when K is lattice congruent to the kite with vertices $(0,1)$, $(1,0)$, $(1,2)$, $(3,1)$ or to the parallelogram with vertices $(0,1)$, $(1,0)$, $(2,2)$, $(3,1)$. See figure 5.5-3.*

```
. o . .         . . o .
o . . o         o . . o
. o . .         . o . .
```

Figure 5.5-3
Only polygons with g=2
and smallest perimeter

Proposition 5.5.6. *If $g = 3$ then $P \geq 3\sqrt{2} + 2\sqrt{5} \approx 8.715$. Equality occurs when and only when K is lattice congruent to the quadrilateral with vertices $(0,1)$, $(1,3)$, $(2,0)$, $(3,1)$ or to the quadrilateral with vertices $(0,1)$, $(1,3)$, $(1,0)$, $(3,1)$. See figure 5.5-4.*

```
. o . .         . o . .
. . . .         . . . .
o . . o         o . . o
. . o .         . o . .
```

Figure 5.5-4
Only polygons with g=3
and smallest perimeter

Proposition 5.5.7. *If $g = 4$ then $P \geq 4\sqrt{5} \approx 8.944$. Equality occurs when and only when K is lattice congruent to the square with vertices $(0,1)$, $(2,0)$, $(1,3)$, $(3,2)$. See figure 5.5-5.*

```
. o . .
. . . o
o . . .
. . o .
```

Figure 5.5-5
Unique polygon with g=4
and smallest perimeter

Proposition 5.5.8. *If $g = 5$ then $P \geq 3\sqrt{5} + \sqrt{13} \approx 10.314$. Equality occurs when and only when K is lattice congruent to the quadrilateral with vertices $(0,2)$, $(2,3)$, $(3,0)$, $(4,2)$. See figure 5.5-6.*

```
. . o . .
o . . . o
. . . . .
. . . o .
```

Figure 5.5-6
Unique polygon with g=5
and smallest perimeter

Proposition 5.5.9. *If $g = 6$ then $P \geq 2\sqrt{5} + 2\sqrt{10} \approx 10.797$. Equality occurs when and only when K is lattice congruent to the parallelogram with vertices $(0,1)$, $(1,3)$, $(3,0)$, $(4,2)$. See figure 5.5-7.*

```
. o . . .
. . . . o
o . . . .
. . . o .
```

Figure 5.5-7
Unique polygon with g=6
and smallest perimeter

Proposition 5.5.10. *If $g = 7$ then $P \geq 2\sqrt{5} + 2\sqrt{13} \approx 11.683$. Equality occurs when and only when K is lattice congruent to the kite with vertices $(0, 1)$, $(2, 0)$, $(2, 4)$, $(4, 1)$ or to the parallelogram with vertices $(0, 2)$, $(1, 4)$, $(3, 0)$, $(4, 2)$. See figure 5.5-8.*

```
. . o . .
. . . . .         . o . . .
. . . . .         . . . . .
o . . . o         o . . . o
. . o . .         . . . . .
                  . . . o .
```

Figure 5.5-8
Only polygons with g=7
and smallest perimeter

Proposition 5.5.11. *If $g = 8$ then $P \geq 4\sqrt{5} + \sqrt{10} \approx 12.107$. Equality occurs when and only when K is lattice congruent to the pentagon with vertices $(0, 1)$, $(1, 3)$, $(3, 4)$, $(4, 1)$, $(2, 0)$. See figure 5.5-9.*

```
. . . o .
. o . . .
. . . . .
o . . . o
. . o . .
```

Figure 5.5-9
Unique polygon with g=8
and smallest perimeter

Proposition 5.5.12. *If $g = 9$ then $P \geq 4\sqrt{10} \approx 12.649$. Equality occurs when and only when K is lattice congruent to the square with coordinates $(0, 1)$, $(1, 4)$, $(4, 3)$, $(3, 0)$. See figure 5.5-10.*

```
. o . . .
. . . . o
. . . . .
o . . . .
. . . o .
```

Figure 5.5-10
Unique polygon with g=9
and smallest perimeter

Proposition 5.5.13. *If $g = 10$ then $P \geq 6\sqrt{5} \approx 13.416$. Equality occurs when and only when K is lattice congruent to the equilateral hexagon with vertices* $(0,1)$, $(1,3)$, $(3,4)$, $(5,3)$, $(4,1)$, $(2,0)$. *See figure 5.5-11.*

```
. . . o . .
. o . . . o
. . . . . .
o . . . o .
. . o . . .
```

Figure 5.5-11
Unique polygon with g=10
and smallest perimeter

A general result can also be obtained, although this result is not best possible.

Theorem 5.5.14. $P > \sqrt{(4g+2)\pi}$.

Proof. From the isoperimetric inequality (Proposition 1.3.1f) we have $P^2 \geq 4\pi A$. We also know that for a convex lattice polygon, $A \geq g + 1/2$ (Proposition 5.2.1). Combining these two results gives

$$P^2 \geq 4\pi A \geq 4\pi(g + \frac{1}{2})$$

or taking square roots,

$$P \geq \sqrt{(4g+2)\pi}.$$

□

Corollary 5.5.15. $P > \sqrt{12g+6}$.

This follows from the fact that $\pi > 3$.

Proposition 5.5.16. $P \geq 2\lceil\sqrt{g}\rceil + 2$.

This will be proven in the next section (corollary 5.6.16).

Proposition 5.5.17. *Let $P(g) = \min\{P(K)|g(K) = g\}$. Then $P(g) \leq 4(\lceil\sqrt{g}\rceil + 1)$.*

Proof. This follows from the Enclosed Square Lemma which guarantees a polygon containing g lattice points and enclosed inside a square of side $\lceil\sqrt{g}\rceil + 1$. The polygon must have perimeter less than the perimeter of this square (since both are convex). □

We can summarize in the following manner.

Theorem 5.5.18. *Let $P(g) = \min\{P(K)|g(K) = g\}$. Then*

a. $P(0) = 2 + \sqrt{2}$.
b. $P(1) = 4\sqrt{2}$.
c. $P(2) = 2\sqrt{2} + 2\sqrt{5}$.
d. $P(3) = 3\sqrt{2} + 2\sqrt{5}$.
e. $P(4) = 4\sqrt{5}$.
f. $P(5) = 3\sqrt{5} + \sqrt{13}$.
g. $P(6) = 2\sqrt{5} + 2\sqrt{10}$.
h. $P(7) = 2\sqrt{5} + 2\sqrt{13}$.
i. $P(8) = 4\sqrt{5} + \sqrt{10}$.

j. $P(9) = 4\sqrt{10}$.
k. $P(10) = 6\sqrt{5}$.
l. $P(g) > \sqrt{12g + 6}$.
m. $2(\lceil\sqrt{g}\rceil + 1) \leq P(g) \leq 4(\lceil\sqrt{g}\rceil + 1)$.

Section 5.6.
Inequalities for Diameter.

Let D denote the diameter of a convex lattice polygon, K.

Proposition 5.6.1 (D unbounded). *For any positive integer, g, there are lattice polygons with g interior lattice points and arbitrarily large diameter.*

Proof. Consider the lattice triangle OAB where $O = (0,0)$, $A = (2g+2,0)$, and $B = (0,2)$. Applying a shear of magnitude k, leaving the x-axis fixed, we find that O and A remain fixed, and B moves to $(2k,2)$. This triangle has g interior lattice points and has a diameter at least as large as $OB = \sqrt{4k^2+4}$. But we can make $\sqrt{4k^2+4}$ arbitrarily large by making k arbitrarily large. Thus, D is unbounded. $\square$

We investigated by computer the relationship between D and g for all convex lattice polygons with $D \leq 10$ and $g \leq 10$. In each case, we found the minimum value of D, which was always less than 5.

Proposition 5.6.2 (effectiveness of search for minimum D). *No lattice polygons with $D \leq 5$ were missed by the computer search.*

This is obvious because we searched all lattice polygons with $D \leq 10$.

The following results were found by computer:

Proposition 5.6.3. *If $g = 0$ then $D \geq \sqrt{2}$. Equality occurs when and only when K is lattice congruent to the square with vertices $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ or to the isosceles right triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$. See figure 5.6-1.*

```
o o      o .
o o      o o
```

Figure 5.6-1
Only polygons with g=0
and smallest diameter

Proposition 5.6.4. *If $g = 1$ then $D \geq 2$. Equality occurs when and only when K is lattice congruent to the diamond with vertices $(0,1)$, $(1,0)$, $(2,1)$, $(1,2)$. See figure 5.6-2.*

```
. o .
o . o
. o .
```

Figure 5.6-2
Unique polygon with g=1
and smallest diameter

Proposition 5.6.5. *If $g = 2$ then $D \geq 3$. Equality occurs when and only when K is lattice congruent to one of the figures shown below.*

```
o . . o      . o-o .
. . . .      o . . o
. o o .      . o-o .
```

Figure 5.6-3

```
Only polygons with g=2
and smallest diameter
```

A pair of circles connected by a dash means that the polygon must contain one or both of these lattice points as vertices.

Proposition 5.6.6. *If $g = 3$ then $D \geq 3$. Equality occurs when and only when K is lattice congruent to the trapezoid with vertices $(0, 1)$, $(1, 0)$, $(1, 3)$, $(3, 1)$. See figure 5.6-4.*

```
. o . .
. . . .
o . . o
. o . .
```

```
       Figure 5.6-4
Unique polygon with g=3
  and smallest diameter
```

Proposition 5.6.7. *If $g = 4$ then $D \geq \sqrt{10}$. Equality occurs when and only when K is lattice congruent to a polygon whose vertices consist of a subset of the vertices of the octagon pictured below. Remove any 0, 1, 2, 3, or 4 of its vertices, but never remove two consecutive vertices.*

```
. o o .
o . . o
o . . o
. o o .
```

```
      Figure 5.6-5
   Polygon with g=4
 and smallest diameter
```

Proposition 5.6.8. *If $g = 5$ then $D \geq 4$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. . o . .
. . . . .      . o x . .
o . . . o      . . . x .
. . . . .      o . . . o
. . o . .      . o-o-o .
```

```
       Figure 5.6-6
 Only polygons with g=5
  and smallest diameter
```

A set of three circles connected by dashes means that some non-empty subset of these three vertices must be vertices of the polygon. A lattice point marked with an x represents an optional point; it may or may not belong to the polygon.

Proposition 5.6.9. *If $g = 6$ then $D \geq 4$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. o . o .
. . . . .
o . . . o
. o-o-o .
```

Figure 5.6-7
Only polygons with g=6
and smallest diameter

A set of three circles connected by dashes means that some non-empty subset of these three vertices must be vertices of the polygon.

Proposition 5.6.10. *If $g = 7$ then $D \geq 4$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. . o . .
. x . x .
. . . . .
o . . . o
. . o . .
```

Figure 5.6-8
Only polygons with g=7
and smallest diameter

A lattice point marked with an x represents an optional point; it may or may not belong to the polygon.

Proposition 5.6.11. *If $g = 8$ then $D \geq 3\sqrt{2}$. Equality can hold as can be seen by figure 5.6-9 in which $g = 8$ and $D = 3\sqrt{2}$.*

```
. o . . .
. . . o .
. . . . .
o . . . o
o o . . .
```

Figure 5.6-9
Polygon with g=8
and smallest diameter

There were too many figures in which equality held to warrant listing them all here.

Proposition 5.6.12. *If $g = 9$ then $D \geq 3\sqrt{2}$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. o . o .
. . . . .
x . . . x
o . . . o
. . o . .
```

Figure 5.6-10
Only polygons with g=9

```
and smallest diameter
```

A lattice point marked with an x represents an optional point; it may or may not belong to the polygon.

Proposition 5.6.13. *If $g = 10$ then $D \geq 5$. Equality can hold as can be seen by figure 5.6-11 in which $g = 10$ and $D = 5$.*

```
. . . o . .
. . . . . .
o . . . . o
. . . . . .
. . . . . .
. . . o . .

      Figure 5.6-11
    Polygon with g=10
  and smallest diameter
```

There were too many figures in which equality held to warrant listing them all here.

A general result can also be obtained, although this result is not best possible.

Lemma 5.6.14. *Let K be a convex body and let $H = \text{hull}(K^\circ \cap Z^2)$. Let K_x and H_x denote the horizontal width of K and H respectively. Then $K_x \geq H_x + 2$.*

This is reasonably obvious after projecting K down to the x-axis.

Theorem 5.6.15. $D \geq \lceil \sqrt{g} \rceil + 1$.

Proof. Let $s = \lceil \sqrt{g} \rceil - 1 = \lfloor \sqrt{g-1} \rfloor$, so that

$$g > s^2. \tag{1}$$

Let H be the convex hull of $K^\circ \cap Z^2$. Let H_x denote the horizontal width of H and let H_y denote the vertical width of H. From (1) we see that either H_x or H_y must be larger than s, for if both H_x and H_y were less than or equal to s, then H could be covered by the square of side s and we would have $g = G(H) \leq s^2$.

Thus we may assume without loss of generality that $H_x > s$. Then by lemma 5.6.14, we would have $K_x \geq H_x + 2$. Thus $K_x > s + 2$. But the diameter of a convex body must be larger than the horizontal width (after all, the diameter is the largest of all the directional widths), so $D \geq s + 2$. □

Corollary 5.6.16. $P \geq 2\lceil \sqrt{g} \rceil + 2$.

This follows from Result 1.3.1d.

Proposition 5.6.17. *There is a convex lattice polygon with $D \leq (\lceil \sqrt{g} \rceil + 1)\sqrt{2}$.*

Proof. By the Enclosed Square Lemma, we can find a lattice polygon with g interior points inside a square of side $t = \lceil \sqrt{g} \rceil + 1$. The diameter of this polygon must be smaller than the diameter of the enclosed square, which is $t\sqrt{2}$. □

Notation. Let $D(g) = \min\{D(K) | g(K) = g\}$.

Proposition 5.6.18. $D(10) = D(11) = D(12) = D(13) = 5$.

Proof. We have already seen that $D(g) \geq \lceil\sqrt{g}\rceil + 1$ so that $D(g) \geq 5$ if $g = 10, 11, 12,$ or 13. To prove that $D(g) = 5$ in these cases, it is only necessary to exhibit a lattice polygon with diameter 5 for these cases. Figure 5.6-11 already established this for $g = 10$. We conclude the proof by exhibiting lattice polygons with diameter 5 in figure 5.6-12. $\square$

```
. o . o . .
. . . . . .
o . . . . o
. . . . . .
. o . . o .
```

```
. o . . o .
. . . . . .
o . . . . o
. . . . . .
. o . . o .
```

```
. . . . o .
. . . . . .
o . . . . o
. . . . . .
. o . . . .
. . . . o .
```

Figure 5.6-12
Polygons with g=11, 12, and 13
and smallest diameter

Proposition 5.6.19. $D(17) = D(18) = D(19) = D(20) = D(21) = 6$.

Proof. Again, we need only exhibit the appropriate lattice polygons with diameter 6. See figure 5.6-13. $\square$

```
. . o . . .
. . . . . .
. . . . . .
o . . . . o
. . . . . .
. . . . . .
. . o . . .
```

```
. . o . o .
. . . . . .
o . . . . .
. . . . . .
. . . . . .
o . . . . o
. . o . . .
```

```
. . o . . .
. . . . . .
. . . . . .
o . . . . o
. . . . . .
. . . . o .
. . o . . .
```

```
. . o . . . .
. o . . . o .
. . . . . . .
. . . . . . .
o . . . . . o
. . . . . . .
. . o . . . .
```

```
. . o . . .
. . . . o .
. . . . . .
o . . . . o
. . . . . .
. . . . o .
. . o . . .
```

Figure 5.6-13
Polygons with g=17, 18, 19, 20, and 21
and smallest diameter

Proposition 5.6.20. $D(26) = D(27) = D(28) = 7$.

Proof. Figure 5.6-14 shows lattice polygons with appropriate g and $D = 7$. Since we have already shown $D(g) \geq 7$ for g in this range, this completes the proof. $\square$

```
. . o . . . .      . . . o . . . .      . . . o . . . .
. . . . . o .      . . . . . o . .      . . . . . o . .
o . . . . . .      . o . . . . . .      . . . . . . . .
. . . . . . .      . . . . . . . .      o . . . . . . o
. . . . . . o      . . . . . . . .      . . . . . . . .
o . . . . . .      o . . . . . . o      . . . . . . . .
. . . . . o .      . o . . . . o .      . o . . . . o .
. . o . . . .      . . . o . . . .      . . . o . . . .
```

Figure 5.6-14
Polygons with g=26, 27, and 28
and smallest diameter

Proposition 5.6.21. $D(37) = 8$.

Proof. Figure 5.6-15 shows a lattice polygon with $g = 37$ and $D = 8$. Since we have already shown $D(37) \geq 8$, this completes the proof. $\square$

```
. . o . . . . . .
. . . . . . . o .
. . . . . . . . .
. . . . . . . . .
o . . . . . . . o
. . . . . . . . .
. o . . . . . o .
. . o . . . . . .
```

Figure 5.6-15
Polygon with g=37
and smallest diameter

We may summarize this data as follows:

Theorem 5.6.22. *Let* $D(g) = \min\{D(K)|g(K) = g\}$. *Then*

a. $D(0) = \sqrt{2}$.
b. $D(1) = 2$.
c. $D(2) = 3$.
d. $D(3) = 3$.
e. $D(4) = \sqrt{10}$.
f. $D(5) = 4$.
g. $D(6) = 4$.
h. $D(7) = 4$.
i. $D(8) = 3\sqrt{2}$.
j. $D(9) = 3\sqrt{2}$.
k. $D(10) = 5$.
l. $D(11) = 5$.
m. $D(12) = 5$.
n. $D(13) = 5$.
o. $D(17) = 6$.
p. $D(18) = 6$.
q. $D(19) = 6$.
r. $D(20) = 6$.
s. $D(21) = 6$.

t. $D(26) = 7$.
u. $D(27) = 7$.
v. $D(28) = 7$.
w. $D(37) = 8$.
x. $\lceil\sqrt{g}\rceil + 1 \le D(g) \le (\lceil\sqrt{g}\rceil + 1)\sqrt{2}$.

Section 5.7.
Inequalities for minimal width.

Let w denote the (minimal) width of a convex lattice polygon, K.

An *altitude* of a polygon is a line through a vertex and perpendicular to a side of the polygon not incident with that vertex. The length of the altitude is the distance from the vertex to the foot of the perpendicular.

Note that the foot of the perpendicular may lie outside the polygon, on the extension of the side to which the altitude is drawn.

Algorithm 5.7.1 (Computation of the width). *The width of a convex polygon can be computed by going to each vertex and finding the length of the largest altitude emenating from that vertex. The width of the polygon is then the smallest of these altitudes.*

The verification of this is straightforward.

This provides an effective means for computing the width of a polygon.

Proposition 5.7.2 (Minimal w). *For any positive integer, g, there are lattice polygons with g interior lattice points and width arbitrarily small.*

This follows from Proposition 2.4.

We investigated by computer the relationship between w and g for all convex lattice polygons with $D \leq 10$ and $g \leq 10$. In each case, we found the maximum value of w, which was always less than 5.

Proposition 5.7.3 (effectiveness of search for maximum w). *No lattice polygons with $w \geq 5$ were missed by the computer search.*

Proof. Consider any polygon with $g \leq 10$ and $w \geq 5$. Then using the fact that $wD \leq 2A$ for all convex bodies (Proposition 1.3.2f), we would have (for $g > 1$)

$$D \leq \frac{2A}{w} \leq \frac{4g+4}{w} \leq \frac{44}{5} < 9.$$

Thus the polygon would have been found in our search since we searched all polygons with $D \leq 10$. (We handle $g = 0$ and $g = 1$ as special cases.) □

The following results were found by computer:

Proposition 5.7.4. *If $g = 0$ then $w \leq \sqrt{2} \approx 1.414$. Equality occurs when and only when K is lattice congruent to the isosceles right triangle with vertices $(0,0)$, $(0,2)$, $(2,0)$. See figure 5.7-1.*

```
o . .
. . .
o . o
```

```
Figure 5.7-1
Unique polygon with g=0
and largest width
```

Proposition 5.7.5. *If $g = 1$ then $w \leq 3\sqrt{2}/2 \approx 2.121$. Equality occurs when and only when K is lattice congruent to the isosceles right triangle with vertices $(0,0)$, $(0,3)$, and $(3,0)$. See figure 5.7-2.*

```
o . . .
. . . .
. . . .
o . . o
```

Figure 5.7-2
Unique polygon with g=1
and largest width

Proposition 5.7.6. *If $g = 2$ then $w \leq 2$. Equality can hold as can be seen by the figure below in which $g = 2$ and $w = 2$.*

```
o . . o
. . . .
o . . o
```

Figure 5.7-3
Polygon with g=2
and largest width

Proposition 5.7.7. *If $g = 3$ then $w \leq 2\sqrt{2} \approx 2.828$. Equality can hold as can be seen by the figure below in which $g = 3$ and $w = 2\sqrt{2}$.*

```
o . . . .
. . . . .
. . . . .
. . . . .
o . . . o
```

Figure 5.7-4
Polygon with g=3
and largest width

Proposition 5.7.8. *If $g = 4$ then $w \leq 3$. Equality can hold as can be seen by the figure below in which $g = 4$ and $w = 3$.*

```
o . . o
. . . .
. . . .
o . . o
```

Figure 5.7-5
Polygon with g=4
and largest width

Proposition 5.7.9. *If $g = 5$ then $w \leq 8\sqrt{5}/5 \approx 3.578$. Equality occurs when and only when K is lattice congruent to the triangle with vertices $(0, 0)$, $(4, 4)$, $(2, 4)$. See figure 5.7-6.*

```
. . o . .
. . . . .
. . . . .
. . . . .
o . . . o
```

Figure 5.7-6
Only polygon with g=5
and largest width

Proposition 5.7.10. *If $g = 6$ then $w \leq 8\sqrt{5}/5 \approx 3.578$. Equality can hold as can be seen by the figure below in which $g = 6$ and $w = 8\sqrt{5}/5$.*

```
. . o . .
. . . . .
. . . . .
o . . . .
o . . . o
```

Figure 5.7-7
Polygon with g=6
and largest width

Proposition 5.7.11. *If $g = 7$ then $w \leq 4$. Equality can hold as can be seen by the figure below in which $g = 7$ and $w = 4$.*

```
. . o . .
. . . . .
o . . . o
. . . . .
o . . . o
```

Figure 5.7-8
Polygon with g=7
and largest width

Proposition 5.7.12. *If $g = 8$ then $w \leq 4$. Equality can hold as can be seen by the figure below in which $g = 8$ and $w = 4$.*

```
o . o . .
. . . . .
. . . . o
. . . . .
o . . . o
```

Figure 5.7-9
Polygon with g=8
and largest width

Proposition 5.7.13. *If $g = 9$ then $w \leq 9\sqrt{5}/5 \approx 4.025$. Equality can hold as can be seen by the figure below in which $g = 9$ and $w = 9\sqrt{5}/5$.*

```
. . o . . .
. . . . . .
. . . . . .
. . . . . o
o . . . . .
o o . . . .
```

Figure 5.7-10
Polygon with g=9
and largest width

Proposition 5.7.14. *If $g = 10$ then $w \leq 2\sqrt{5} \approx 4.472$. Equality can hold as can be seen by the figure below in which $g = 10$ and $w = 2\sqrt{5}$.*

```
. . o . . .
. . . o . .
. . . . . .
. . . . . .
. . . . . .
o . . . . o
```

Figure 5.7-11
Polygon with g=10
and largest width

A general result can also be obtained, although this result is not best possible.

Theorem 5.7.15. $w \leq \sqrt{2(g+1)\sqrt{3}}$.

Proof. This is clear if $g = 0$. If $g > 0$, then we know that $A \leq 2g + 2$. We can combine this with the inequality $w^2 \leq A\sqrt{3}$ which is true for all convex bodies (proposition 1.3.1h), to get $w^2 \leq (2g + 2)\sqrt{3}$. Taking square roots of both sides gives us the desired result. □

Proposition 5.7.16. *There are lattice polygons with $w \leq \lceil\sqrt{g}\rceil + 1$.*

Proof. By the Enclosed Square Lemma, we can find a lattice polygon with g interior points inside a square of side $t = \lceil\sqrt{g}\rceil + 1$. The width of this polygon must be smaller than t, the width of the square. □

We may summarize this data as follows:

Theorem 5.7.17. *Let $w(g) = \max\{w(K)|g(K) = g\}$. Then*

a. $w(0) = \sqrt{2}$.
b. $w(1) = 3\sqrt{2}/2$.
c. $w(2) = 2$.
d. $w(3) = 2\sqrt{2}$.
e. $w(4) = 3$.
f. $w(5) = 8\sqrt{5}/5$.
g. $w(6) = 8\sqrt{5}/5$.
h. $w(7) = 4$.
i. $w(8) = 4$.
j. $w(9) = 9\sqrt{5}/5$.
k. $w(10) = 2\sqrt{5}$.

l. $\lceil\sqrt{g}\rceil + 1 \le w(g) \le \sqrt{2(g+1)\sqrt{3}}.$

Note the surprising fact that $w(g)$ is not monotonic, since

$$w(1) > w(2) < w(3).$$

We mention some known generalizations to higher dimensions. Let K be a convex body in E^n. Let g denote the number of lattice points in the interior of K. Let w be the (minimal) width of K and let r be the inradius of K. Let D be the diameter of K.

Result 5.7.18. *Let* $w(n,0) = \max\{w(K)|K \subset E^n, g(K) = 0\}$. *Then*

a. $\sqrt{n} \le w(n,0)$.

b. If n is odd, then $w(n,0) \le n$.

c. If n is even, then $w(n,0) \le (n+1)\sqrt{\frac{n}{n+2}}$.

d. $w(n,0) > (\sqrt{2}+1)(\sqrt{n+1} - \beta)$ *where* $\beta = 3\sqrt{2} - 4 + 7\sqrt{\frac{1}{3}} - 4\sqrt{\frac{2}{3}}$.

e. There are constants λ_n and μ_n (independent of K), with $1/\sqrt{2} \le \lambda_n \le 1$ and $1/\sqrt{n} \le \mu_n \le 1$ such that

$$(\lambda_n \frac{w(K)}{w(n-1,0)} - 1)(\mu_n D(K) - 1) \le 1.$$

Reference. McMullen and Wills [72].

Conjecture (McMullen and Wills).

a. $w(n,0)$ is achieved by a regular simplex.

b. $\lambda_n = 1 = \mu_n$ in the above result.

Result 5.7.19.

a. If K is a centrally symmetric convex body, and $g = 1$, then $V \le 2^n$.

b. If K is a centrally symmetric convex body, and $g = 1$, then $G \le 3^n$, where G is the number of lattice points covered by K. If $g = 1$ and $V = 2^n$, then K is a convex polytope and it has at most $2^{n+1} - 2$ facets.

Reference. Minkowski [75] and Zaks, Perles, and Wells [128].

Result 5.7.20. *If a is a positive real number, then we define* $g(a,n) = \min\{g(K)|w(K) > a\}$. *Then*

a.
$$g(\frac{2+\sqrt{3}}{2}, 2) = 1.$$

b.
$$g(a,2) \ge \lfloor \frac{2a}{2+\sqrt{3}} \rfloor^2.$$

c.
$$g(a,2) \le \lfloor \frac{a^2}{\sqrt{3}} \rfloor.$$

Reference. Scott [101].

Result 5.7.21. *If a is a positive real number and K is centrally symmetric about the origin, we define* $g_0(a,n) = \min\{g(K)|w(K) > a\}$. *Then* $g_0(2,n) = 2n+1$.

Reference. Scott [101].

Result 5.7.22. *A centrally symmetric convex body $K \subset E^n$ centered at the origin, and of width a, contains the n-dimensional ball centered at the origin of radius $a/2$.*

Reference. Scott [101].

Section 6
Relationship between v and g

Let K be a convex lattice pentagon with v vertices and g interior lattice points. In 1980, Arkinstall proved the Lattice Pentagon Theorem, which says that any lattice pentagon must contain an interior lattice point. In other words, $v = 5 \Rightarrow g \geq 1$. In this section we will investigate further the relationship between v and g for lattice polygons.

Section 6.1.
g versus v.

If we fix v, g can get arbitrarily large. For given any polygon, K, with v vertices, we can expand it by any amount. The resulting polygon has the same number of vertices, but g can get arbitrarily large. In other words, for a fixed v,

$$\sup\{g(K)|v(K) = v\} = \infty.$$

A more interesting problem is to find the minimum value that g can have when we fix v.

Proposition 6.1.1. $v = 3 \Rightarrow g \geq 0$. *Equality occurs when and only when K is lattice equivalent to either TRIANG$(2,2)$ or TRIANG$(p,1)$ for some positive integer p.*

Clearly g can get as small as 0 as can be seen by TRIANG(2,2). The fact that the figures listed are the only ones in which equality holds will be proven in section 9 (Theorem 9.1.1).

The Lattice Trapezium Theorem. *If $v = 4$ and K has no pair of parallel edges, then $g \geq 1$. In other words, a lattice trapezium must contain a lattice point.*

Proof. (Arkinstall [3]). Let the quadrilateral be called $ABCD$. It is not possible for the sum of every pair of adjacent angles to be π, for if that were the case, then the sum of all 4 pairs would be larger than 4π contradicting the fact that twice the sum of the angles of a quadrilateral is exactly 4π. So we may assume that $\angle B + \angle C > \pi$. We may also assume without loss of generality that vertex D is closer than A from the line BC (otherwise perform a reflection about the perpendicular bisector of BC). Locate point X such that $BCDX$ is a parallelogram.

Since $\angle XBC + \angle BCD = \pi$ but $\angle ABC + \angle BCD > \pi$, it must be true that ray BX lies inside angle ABC. Since A is no closer than D to BC and since AD is not parallel to BC, we can conclude that X lies inside $ABCD$. Since B, C, and D are lattice points, so too must X be a lattice point and we have therefore found a lattice point inside $ABCD$. $\square$

The Lattice Pentagon Theorem. *If $v = 5$ then $g \geq 1$.*

Proof. (Arkinstall [3]). Let the pentagon be $ABCDE$. Proceeding exactly as before, assume $\angle ABC + \angle BCD > \pi$ and D is no further than A from BC. Locate X as before to make $XBCD$ a parallelogram. In this case, X might lie on AD, but in any case, we have found a lattice point, X, inside the pentagon. $\square$

Corollary 6.1.2. *If $v \geq 5$ then $g \geq 1$.*

If $v > 5$, removing any vertex yields a convex lattice pentagon, and thus the polygon contains an interior lattice point.

Proposition 6.1.3. *If $v = 5$ then $g \geq 1$. Equality holds when and only when K is lattice equivalent to one of the following 5 pentagons:*

```
. o .        . o .        o o .
o . o        o . o        . . o
o . o        o o .        o . o
```

`Figure 6.1-1`
`Only lattice polygons with v=5`
`and smallest g`

This result is proven in section 9 (Proposition 9.2.3).

The Central Hexagon Theorem (Arkinstall). *If $v = 6$ and $g = 1$, then K is lattice equivalent to the centrally symmetric hexagon with vertices at $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 0)$, $(-1, -1)$, and $(0, -1)$ as shown in figure 6.1-2.*

The following proof is due to Arkinstall [3].

Proof. Let the hexagon be called $ABCDEF$, and call its unique interior lattice point, P. If P does not lie on diagonal AD, then it must lie inside one of the two quadrilaterals formed by AD, say P lies inside $ADEF$. But then $ABCDP$ would be a convex lattice pentagon and so it would have to contain an interior lattice point, Q, by the Lattice Pentagon Theorem. This contradicts the fact that $ABCDEF$ contains just one interior lattice point.

The same reasoning shows that P lies on BE and CF. Since all the lattice points on a line are equally spaced, this shows that P must be the common midpoint of segments AD, BE, and CF. These diagonals must be parallel to the edges of the hexagon, for otherwise, they would cut off a lattice trapezium and The Lattice Trapezium Theorem would yield another lattice point inside the hexagon.

By the x-axis Lemma, we may find an integral unimodular affine transformation that maps P to the origin, D to $(1, 0)$, and maps E and F above the x-axis. Then A maps to $(-1, 0)$. Now DE cannot contain a lattice point, Q in its relative interior, because then R, the fourth point of parallelogram $PDQR$ would lie on PF and be another lattice point in the interior of the hexagon. Thus $\triangle PDE$ has $g = 0$ and $b = 3$, so by Pick's Formula, it has $A = 1$ implying that the height of E above the x-axis is 1. We may therefore apply a shear, leaving the x-axis fixed and moving E into $(1, 1)$. This forces B to $(-1, -1)$ and since $CF \parallel DE$, we find that F goes to $(0, 1)$. The hexagon is therefore lattice equivalent to the one shown in figure 6.1-2. $\square$

Corollary 6.1.4. $v = 6 \Rightarrow g \geq 1$. *Equality occurs when and only when K is lattice equivalent to the centrally symmetric hexagon shown in figure 6.1-2.*

```
. o o
o . o
o o .
```

`Figure 6.1-2`
`Unique lattice polygon with v=6`
`and smallest g`

Arkinstall showed that $v = 7 \Rightarrow g \geq 2$. We will give a slightly simpler proof and then show the best possible result, that $v = 7 \Rightarrow g \geq 4$.

Lemma 6.1.5. *$v = 7 \Rightarrow g \geq 2$.*

Proof. Let $ABCDEFG$ be a convex lattice heptagon. Then $ABCDE$ is a convex lattice pentagon so must contain an interior lattice point, X, by the Lattice Pentagon Theorem. Then $AXEFG$ is another convex lattice pentagon, so it must contain an interior lattice point, Y. □

Lemma 6.1.6. *Let K be a convex lattice polygon. If $v = 7$ and $g = 3$ then the line joining any two interior lattice points must pass through two vertices of K.*

Proof. Let the two interior lattice points be X and Y. The line XY divides the heptagon into two pieces. If one of these pieces contains exactly 1 or 2 vertices (not on XY) and XY doesn't pass through 2 vertices, this would be a contradiction, for in the other piece, we would be able to create a heptagon, thereby finding another 2 interior lattice points by lemma 6.1.5. If one of these pieces contains exactly 3 vertices (not on XY) and XY doesn't pass through 2 vertices, this would also be a contradiction, for we would find two pentagons present, one in each piece, thereby finding another 2 interior lattice points by the Lattice Pentagon Theorem. The same holds if both pieces contain exactly 3 vertices and XY went through 2 other vertices. This covers all cases. □

Proposition 6.1.7. *$v = 7 \Rightarrow g \geq 4$. Equality can hold as can be seen by the heptagon in figure 6.1-3 in which $v = 7$ and $g = 4$.*

```
. o . .     . o o .     . o o .
o . . o     . . . o     . . . o
o . . o     o . . o     o . . o
. o o .     o . o .     o o . .
```

Figure 6.1-3
Lattice polygons with v=7
and smallest g

Proof. Let $ABCDEFG$ be a convex lattice heptagon. There must be two interior lattice points, X and Y, by lemma 6.1.5. By lemma 6.1.6, line XY passes through two vertices, say P and Q. Of the other 5 vertices of the heptagon, at least 3 of them must fall on one side of PQ. Call these A, B and C, in order, with A nearest to P. Pentagon $XYCBA$ must contain a lattice point. Call it Z. By lemma 6.1.6, XZ must pass through A, B, or C. If it passes through A, then $XZBCQ$ would be another pentagon and we would be done. Similarly, if it passes through C. Thus XZ must pass through B. In the same manner, we find that YZ must also pass through B. This is a contradiction since X and Y are distinct points. □

There are lots of figures for which equality holds, so we will not bother to list them all here. A few examples are shown above in figure 6.1-3.

Proposition 6.1.8. *If $v \geq 7$, then the interior lattice points of K can not colline.*

Proof. Suppose $v \geq 7$ and that all the lattice points interior to K lie on a line L. Since $v \geq 7 \Rightarrow g > 2$, there are at least two lattice points, say P and Q on line L inside K. L can meet K at at most 2 points, so there are at least 5 vertices of K that do not lie on L. L divides the plane into two regions, and we have 5 points, so at least 3 of these vertices, say A, B, and C, lie in one of the regions. Then $ABCPQ$ would be a convex lattice pentagon and thus would have an interior lattice point by the Lattice Pentagon Theorem. The existence of this point contradicts the fact that all the lattice points interior to K lie on L. $\square$

The Central Octagon Theorem. *If K is a lattice polygon with $v = 8$ and $g = 4$, then K is lattice equivalent to the centrally symmetric octagon shown in figure 6.1-4.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 6.1-4
Unique lattice polygon with v=8
and smallest g

Proof. Let the octagon be $ABCDEFGH$. Draw in AD. Quadrilateral $ABCD$ can't contain an interior lattice point, P, for then $APDEFGH$ would be a 7-gon and we would thus have an additional 4 lattice points interior to K. Therefore $AD \parallel BC$ by the Lattice Trapezium Theorem. Similarly, $HC \parallel AB$.

Let diagonals AD and CH meet at point P. We have just shown that $ABCP$ is a parallelogram. Since points A, B, and C are lattice points, it follows that P must be a lattice point.

In a similar manner, we find the other three interior lattice points, Q, R, and S and see that they form a parallelogram. A suitable integral unimodular affine transformation transforms this parallelogram into a square. This transformation also forces each vertex of the octagon to be in fixed positions on the extensions of the sides of the square; so we see that the resulting octagon is lattice equivalent to the one shown. $\square$

Corollary 6.1.9. $v = 8 \Rightarrow g \geq 4$. *Equality occurs when and only when K is lattice equivalent to the centrally symmetric octagon shown in figure 6.1-4.*

Proof. If $v = 8$, then remove one vertex to get a convex lattice polygon with $v = 7$ which implies there are at least 4 interior lattice points. $\square$

Notation. Let $g(v) = \min\{g(K)|v(K) = v\}$.

Thus, we have already shown that $g(3) = 0$, $g(4) = 0$, $g(5) = 1$, $g(6) = 1$, $g(7) = 4$, and $g(8) = 4$. We wish now to study the properties of $g(v)$.

Note that there should be no confusion between this function, $g(v)$ and the lattice point counting function, $g(K)$ since the domain of $g(v)$ is Z^+ whereas the domain of $g(K)$ is the set of convex bodies in the plane.

Proposition 6.1.10. *$g(v)$ is monotone.*

Proof. Let K be any v-gon. Remove one vertex from K to get a $(v-1)$-gon called K^*. K^* has at least $g(v-1)$ interior lattice points. Since each K has at least $g(v-1)$ interior lattice points, so must the the min over all K have at least $g(v-1)$ interior lattice points. Thus $g(v) \geq g(v-1)$. $\square$

Proposition 6.1.11. *If $v \geq 5$, then $g(v+2) \geq g(v) + 1$.*

Proof. Let $A_1A_2A_3A_4A_5 \ldots A_{v+2}$ be a convex $(v+2)$-gon with $v \geq 5$. Polygon $A_1A_2A_3A_4A_5$ is a convex lattice pentagon, so it must contain a lattice point, P, in its interior. Polygon $A_1PA_5A_6 \ldots A_{v+2}$ is a convex lattice v-gon, so it must contain $g(v)$ additional lattice points. Thus K contains at least $g(v) + 1$ interior lattice points. $\square$

Proposition 6.1.12. *If $v \geq 7$, then $g(v+2) \geq g(v)+2$.*

Proof. Let $A_1A_2A_3A_4A_5A_6 \ldots A_{v+2}$ be a convex $(v+2)$-gon with $v \geq 7$. Polygon $A_1A_2A_3A_4A_5$ is a convex lattice pentagon, so it must contain a lattice point, P, in its interior. Polygon $A_1PA_5A_6 \ldots A_{v+2}$ is a convex lattice v-gon, so it must contain $g(v)$ additional interior lattice points, P_i.

Since $v \geq 7$, quadrilaterals $A_{v+2}A_1A_2A_3$ and $A_3A_4A_5A_6$ meet only at point A_3. Thus, P cannot lie inside (or on) both of these quadrilaterals. Without loss of generality, assume that P does not lie inside or on $A_{v+2}A_1A_2A_3$.

If A_3P meets A_4A_5, then $A_3PA_5A_6 \ldots A_vA_{v+1}A_{v+2}A_1A_2$ is a convex lattice $(v+2)$-gon and must contain at least $g(v)+1$ more interior lattice points, by Proposition 6.1.11. Thus $A_1A_2A_3A_4A_5 \ldots A_{v+2}$ would contain $g(v)+2$ interior lattice points and we would be done.

The only other case is when the line A_1P meets the polygonal arc $A_5A_6A_7 \ldots A_vA_{v+1}A_{v+2}$. Let A_3P meet this polygonal arc at point Q. We said that P does not lie on quadrilateral $A_{v+2}A_1A_2A_3$, so Q cannot coincide with A_{v+2}. Let R be the polygonal region bounded by this polygonal arc and PA_1 and PQ. If region R contains none of the P_i in its interior, then $A_1A_2A_3PA_{v+2}$ is a convex lattice pentagon which must contain a lattice point. This would bring the total number of lattice points in K up to $g(v)+2$ and we would be done. If some P_i is in the interior of region R, label the P_i so that P_1 is the one closest to line PA_1. Then triangle PA_1P_1 contains none of the other P_i (otherwise P_1 wouldn't be the closest point to PA_1). Thus $A_1A_2A_3PP_1$ would be a convex lattice pentagon. It would have to contain another lattice point, and we are done. $\square$

We now present a second proof that illustrates another useful technique.

Proof. Let K be a convex lattice polygon with $v+2$ vertices with $v \geq 7$. Let H be the convex hull of the interior lattice points of K. Since $v \geq 7$, $\mathrm{card}(G(H)) \geq 2$. Since H is a polygon (or a line segment), there are two lattice points, P and Q on the boundary of H that form a support line of H. (PQ is a side or part of a side of H.) PQ divides K into two pieces. Let K_1 be the piece (not including the line PQ) that does not contain the interior of H. K_1 can't contain 3 or more vertices of K, for if it contained 3 vertices, A, B, and C, then $ABCPQ$ would be a convex lattice pentagon and, by the Lattice Pentagon Theorem, would contain a lattice point, contradicting the fact that K_1 contains no lattice points in its interior.

So K_1 contains 2 or fewer vertices of K. Throwing away these 2 or fewer vertices, and replacing them with P and Q, we obtain (with the remainder of K) a convex lattice polygon with at least v vertices. This polygon has $g(v)$ interior lattice points. Those, plus P and Q, show that $g(v+2) \geq g(v)+2$. $\square$

Corollary 6.1.13. $v = 9 \Rightarrow g \geq 6$.

This follows immediately from the fact that $g(7) = 4$. This result is not best possible.

Proposition 6.1.14. *If $n \geq 4$ then $g(2n-1) \geq 2n-4$ and $g(2n) \geq 2n-4$.*

This follows by induction on n and the fact that $g(7) = g(8) = 4$.

Corollary 6.1.15. *If $v \geq 7$, then $g(v) \geq 2\lfloor \frac{v-4}{2} \rfloor$.*

Corollary 6.1.16. *If $v \geq 7$, then $g(v) \geq v-5$.*

Definition. Let K be a convex body in the plane. Then $H(K)$ is the convex hull of lattice points interior to K. $H(K)$ is called the *interior hull* of K.

This will frequently be denoted by just H, if K is fixed. In other words,

$$H(K) = \partial\big(\text{hull}(K^\circ \cap Z^2)\big).$$

Loosely speaking, H is the largest convex lattice polygon contained within K. Note, however, that H might degenerate into a line segment or the null set.

Lemma 6.1.17. *If K is a convex lattice polygon and $G(K) \geq 5$, then there is a lattice point, P, in or on K that is not a vertex of K.*

Proof. If all the lattice points in or on K were vertices, then we would have a convex lattice pentagon. This would imply another lattice point interior to K by the Lattice Pentagon Theorem. □

Theorem 6.1.18 (The Interior Hull Inequality). *Let K be a convex lattice polygon and let $H = H(K)$. If $v(K) \geq 9$, then $2v(K) \leq 3b(H)$.*

Proof. Since $v(K) \geq 9 \Rightarrow g(K) \geq 5$, we must have $G(H) = g(K) \geq 5$. Thus, by Lemma 6.1.17, we may find some lattice point, P, in or on H, that is not an extreme point of H (i.e. not a vertex of H). Also, the interior lattice points do not all colline because $v \geq 7$ (see Proposition 6.1.8).

If we draw rays from P to each of the lattice points on the boundary of H, the angle between these rays contains no other lattice points of H and will never be larger than π. We have thus divided H into at most $b(H)$ wedges. Note that P might be strictly interior to H or it might be on the boundary of H (but it is not a vertex of H).

For purposes of this proof, define an *element* of K to be either a vertex of K or an edge of K. Consider any wedge, with rays PX and PY where P, X, and Y are lattice points of H. Let Q be the lattice point in H and strictly inside the wedge that is closest to segment XY. If there are no such points, let Q be P. Let the cone formed by QX and QY be known as the extended wedge. The extended wedge contains no interior lattice points of K. There cannot be two vertices of K, say A and B, strictly inside the wedge, for then $ABXQY$ would be a convex lattice pentagon implying that the wedge contained another interior lattice point.

There cannot be two edges of K, say AB and BC inside the wedge for then $ABCXY$ would be a convex lattice pentagon implying that the wedge contained another interior lattice point.

Thus each wedge has at most 4 elements. Thus the total number of elements in all can't be more than 3 times the number of wedges, or not more than $3b(H)$. Each element is counted twice by this process. The total number of elements is just $2v(K)$, so $3b(H) \geq 2v(K)$. □

Lemma. Let K be a convex lattice polygon with v vertices and g interior lattice points. Let H be the interior hull of K. If $g(K) = g(v-2) + 2$, then H is lean.

Proof. If H were not lean, then there would be some support line of H, L, containing 3 or more lattice points of ∂H, say P, Q, and R. L divides K into two pieces. Let K_1 be the piece that does not contain any portion of H in its interior and let K_2 be the other piece. Then K_1 must have fewer than 3 vertices (not counting any on L). For if it had 3 vertices, A, B, and C, then $ABCRP$ would be a convex lattice pentagon and K_1 would contain a lattice point in its interior.

Thus K_2 must contain at least $v-2$ vertices (including any that might be endpoints of L). But any $(v-2)$-gon must contain at least $g(v-2)$ interior lattice points. Those, plus the 3 on L show that K contains at least $3 + g(v-2)$ lattice points, a contradiction. Thus H is lean. □

Theorem 6.1.19. $v \geq 2n+1 \Rightarrow g \geq 3n-5$.

Proof. We proceed by induction. The theorem has already been shown to be true if $n=2$ or $n=3$. (It is trivially true for $n=0$ and $n=1$.) So suppose it is true for all integers smaller than n, we will now show it is true for n ($n \geq 4$).

Let K be a convex lattice polygon with $2n+1$ sides with $n > 1$. By the induction hypothesis, we know that $g(2n-1) \geq 3(n-1)-5 = 3n-8$. Hence, by Proposition 6.1.12, $g(2n+1) \geq 3n-6$.

Let H be the interior hull of K. If $G(H) = 3n-5$, then we are done, and since $G(H) = g(2n+1) \geq 3n-6$, we may assume that $G(H) = 3n-6$.

By the lemma, H is lean. Thus $v(H) = b(H)$.

Claim. $v(H) < \lfloor \frac{4n+4}{3} \rfloor$.

Proof. Suppose H has $\lfloor \frac{4n+4}{3} \rfloor$ or more vertices. There are three cases to consider, depending on the remainder when n is divided by 3. We will reach a contradiction by showing that in each case, $G(H) > 3n-6$.

We first note that $v(H) > n+1$. This is because $\lfloor \frac{4n+4}{3} \rfloor \geq \frac{4n+4}{3} - 1 = \frac{4n+1}{3} > n+1$ (since $n > 2$).

Case 1: $n = 3k$.

In this case, $v(H)$ is at least $\lfloor \frac{12k+4}{3} \rfloor = 4k+1$. But $4k+1 < 2n+1$, so by the inductive hypothesis, $g(H) \geq 6k-5 = 2n-5$. Thus $G(H) \geq v(H) + g(H) > (n+1) + (2n-5) = 3n-4 > 3n-6$, the desired contradiction.

Case 2: $n = 3k+1$.

In this case, $v(H)$ is at least $\lfloor \frac{12k+8}{3} \rfloor = 4k+2 > 4k+1$. But $4k+1 < 2n+1$, so by the inductive hypothesis, $g(H) \geq 6k-5 = 2n-7$. Thus $G(H) \geq v(H)+g(H) > (n+1) + (2n-7) = 3n-6$, the desired contradiction.

Case 3: $n = 3k+2$.

In this case, $v(H)$ is at least $\lfloor \frac{12k+12}{3} \rfloor = 4k+4 > 4k+3$. But $4k+3 < 2n+1$, so by the inductive hypothesis, $g(H) \geq 6k-2 = 2n-6$. Thus $G(H) \geq v(H)+g(H) > (n+1) + (2n-6) = 3n-5 > 3n-6$, the desired contradiction.

This proves our claim. ☐

We have just shown that $v(H) = b(H) < \lfloor \frac{4n+4}{3} \rfloor$ or $b(H) \leq \lfloor \frac{4n+4}{3} \rfloor - 1$. By the Interior Hull Inequality, we have

$$v(K) \leq \frac{3}{2} b(H) \leq \frac{3}{2} \left\lfloor \frac{4n+4}{3} \right\rfloor - \frac{3}{2} \leq \frac{3}{2}\left(\frac{4n+4}{3}\right) - \frac{3}{2} = 2n + \frac{1}{2} < 2n+1$$

contradicting the fact that $v(K) = 2n+1$. Hence our assumption that $g(K) = 3n-6$ is incorrect, and we must have $g(K) \geq 3n-5$. ☐

Corollary 6.1.20. $g(2n+1) \geq 3n-5$.

Corollary 6.1.21. $g(2n+2) \geq 3n-5$.

Corollary 6.1.22. $g(v) \geq 3\lfloor \frac{v-1}{2} \rfloor - 5$.

This comes from combining the previous two inequalities. Also note that the result is trivially true if $v=3$ or $v=4$.

Proposition 6.1.23. $v = 9 \Rightarrow g \geq 7$. *Equality holds when and only when K is lattice equivalent to the nonagon shown in figure 6.1-5.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

Figure 6.1-5
Unique lattice polygon with v=9
and smallest g

Proof. Let $v = 9$ in Corollary 6.22. That the pictured polygon is unique comes from a computer study. $\square$

Proposition 6.1.24. $v = 10 \Rightarrow g \geq 10$. *Equality holds when and only when K is lattice equivalent to the decagon shown in figure 6.1-6.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

Figure 6.1-6
Unique lattice polygon with v=10
and smallest g

This unique polygon was discovered by computer search.

To find an upper bound for $g(v)$, we need only exhibit a polygon with v vertices and g interior lattice points.

Proposition 6.1.25. *There is a lattice polygon with $v = 2n$ and $g = \binom{n}{3}$.*

Proof. Let $A_1 = (0, 0)$ and $B_1 = (1, 0)$. We define A_k recursively by $A_{k+1} = A_k + (k + 1, 1)$ for $k = 1, 2, \ldots, n - 1$. That is, to get to A_{k+1} from B_k, you move right $k + 1$ units and then up 1 unit. We define B_k recursively by saying that $B_{k+1} = B_k + (n + 1 - k, 1)$ for $k = 1, 2, \ldots, n - 1$.

This polygon is shown in figure 6.1-7 for the case $n = 5$.

```
. . . . . . . . . . o o
. . . . . . o . . . o .
. . . o . . . . o . . .
. o . . . o . . . . . .
o o . . . . . . . . . .
```

Figure 6.1-7

This polygon has $2n$ vertices, all at lattice points.

Since the abscissae increase in steps of $1, 2, \ldots, n - 1$ for both the A_k and the B_k, it follows that A_n is one unit to the left of B_n since A_1 was one unit to the left of B_1. This fact, plus the way the slopes of the sides were chosen, assures us that the polygon is convex.

We will now count the number of lattice points interior to this polygon. The polygon has a height of $n - 1$, so there are $n - 2$ horizontal lines upon which interior lattice points may lie. They lie on the line segments $A_kB_k, k = 2, 3, \ldots n - 1$. It is easy to sum up the abscissae to find

$$A_k = (\sum_{i=1}^{k-1} i, k-1)$$

and

$$B_k = (1 + \sum_{i=1}^{k-1} (n-i), k-1)$$

so that the distance from A_k to B_k is

$$\begin{aligned} 1 + \sum_{i=1}^{k-1}(n-i) - \sum_{i=1}^{k-1} i &= 1 + \sum_{i=1}^{k-1}(n-2i) \\ &= 1 + \sum_{i=1}^{k-1} n - 2\sum_{i=1}^{k-1} i \\ &= 1 + n(k-1) - k(k-1) = 1 + (n-k)(k-1). \end{aligned}$$

Thus the number of lattice points on this line segment and inside K is just $(n-k)(k-1)$. The total number of lattice points inside K is therefore

$$\begin{aligned} \sum_{k=1}^{n-1}(n-k)(k-1) &= \sum_{k=1}^{n-1}(n+1)k - \sum_{k=1}^{n-1} k^2 - \sum_{k=1}^{n-1} n \\ &= (n+1)\frac{n(n-1)}{2} - \frac{(n-1)n(2n-1)}{6} - (n-1)n \\ &= \frac{n(n-1)(n-2)}{6}. \end{aligned}$$

(We could start summing at $k=1$ because we know that A_1B_1 contributes 0 to the sum.) This final answer shows that $g = \binom{n}{3}$ as claimed. □

Corollary 6.1.26. *There is a lattice polygon with $v = 2n-1$ and $g = \binom{n}{3}$.*

Proof. Vertex A_1 can be removed from the polygon exhibited above without changing the number of interior lattice points. □

Corollary 6.1.27. $g(2n) \leq n(n-1)(n-2)/6$ *and* $g(2n-1) \leq n(n-1)(n-2)/6$.

Corollary 6.1.28. $g(n) \leq \binom{\lceil n/2 \rceil}{3}$.

Corollary 6.1.29. $g(10) \leq 10$, $g(11) \leq 20$, $g(12) \leq 20$, $g(13) \leq 35$, $g(14) \leq 35$, $g(15) \leq 56$, $g(16) \leq 56$, *and* $g(17) \leq 84$.

Proposition 6.1.30. *For any n, there exists a lean lattice polygon with n vertices.*

The polygon we just exhibited is lean.

Proposition 6.1.31. $g(11) \leq 17$, $g(12) \leq 19$, $g(13) \leq 27$, $g(14) \leq 34$, *and* $g(15) \leq 48$.

We need only exhibit the appropriate polygon. (See figures 6.1-8 through 6.1-12.)

```
. . . . . . o o .
. . . . o . . . o
. . . . . . . . o
. o . . . . . o .
o . . . . o . . .
o o . . . . . . .
```

Figure 6.1-8
Lattice polygon with v=11 and g=17

```
. . . . o o .
. . o . . . o
. o . . . . o
. . . . . . .
o . . . . o .
o . . . o . .
. o o . . . .
```

Figure 6.1-9
Lattice polygon with v=12 and g=19

```
. . . . . . . . . o o .
. . . . . . o . . . . o
. . . . o . . . . . . o
. . . . . . . . . . o .
. o . . . . . . o . . .
o . . . . o . . . . . .
o o . . . . . . . . . .
```

Figure 6.1-10
Lattice polygon with v=13 and g=34

```
. . . . . . . o o .
. . . . o . . . . o
. . o . . . . . . o
. o . . . . . . . .
. . . . . . . . o .
o . . . . . . o . .
o . . . . o . . . .
. o o . . . . . . .
```

Figure 6.1-11
Lattice polygon with v=14 and g=34

```
. . . . . . . . . . . . . o . o .
. . . . . . . . . o . . . . . . o
. . . . . . o . . . . . . . . . o
. . . . o . . . . . . . . . . o .
. . . . . . . . . . . . . o . . .
. o . . . . . . . . o . . . . . .
o . . . . . o . . . . . . . . . .
o o . . . . . . . . . . . . . . .
```

Figure 6.1-12
Lattice polygon with v=15 and g=48

```
. . . . . . . . . . . o o .
. . . . . . . o . . . . . o
. . . . o . . . . . . . . o
. . o . . . . . . . . . . .
. o . . . . . . . . . . o .
. . . . . . . . . . . o . .
o . . . . . . . . o . . . .
o . . . . . o . . . . . . .
. o o . . . . . . . . . . .
```

Figure 6.1-13
Lattice polygon with v=16 and g=56

We may summarize as follows:

Theorem 6.1.32. *Let* $g(v) = \inf\{g(K) | v(K) = v\}$*. Then*

a. $g(3) = 0$.
b. $g(4) = 0$.
c. $g(5) = 1$.
d. $g(6) = 1$.
e. $g(7) = 4$.
f. $g(8) = 4$.
g. $g(9) = 7$.
h. $g(10) = 10$.
i. $g(11) \leq 17$.
j. $g(12) \leq 19$.
k. $g(13) \leq 27$.
l. $g(14) \leq 34$.
m. $g(15) \leq 48$.
n. $g(16) \leq 56$.
o. $g(v) \geq g(v-2) + 2$ *for* $v \geq 9$.
p. $g(2n+1) \geq 3n - 5$.
q. $3\lfloor \frac{v-1}{2} \rfloor - 5 \leq g(v) \leq \binom{\lceil v/2 \rceil}{3}$.

Section 6.2.
v versus g.

We now look at the related problem of finding the bounds on v for any given value of g.

Proposition 6.2.1.

$$\inf\{v(K)|g(K)=g\}=3.$$

This is an immediate consequence of Proposition 3.4.

A more interesting problem is to find the maximum value that v can have when we fix g.

Proposition 6.2.2. $g=0 \Rightarrow v \leq 4$.

Proof. If v were greater than 4, there would be an interior point by the Lattice Pentagon Theorem. □

Proposition 6.2.3. $g=1 \Rightarrow v \leq 6$. *Equality occurs when and only when K is lattice equivalent to the centrally symmetric hexagon shown in figure 6.2-1.*

```
. o o
o . o
o o .
```

```
Figure 6.2-1
Unique lattice polygon with g=1
and largest v
```

Proof. $v \geq 7 \Rightarrow g \geq 4 > 1$. Equality is determined by the Central Hexagon Theorem. □

Proposition 6.2.4. $g=2 \Rightarrow v \leq 6$. *Equality can hold as can be seen by figure 6.2-2 in which $v=6$ and $g=2$.*

```
. o o .      . o o .      . o . o
o . . o      o . . o      o . . o
. o o .      o . o .      o . o .
```

```
Figure 6.2-2
Some lattice polygons with g=2
and largest v
```

Proof. $v \geq 7 \Rightarrow g \geq 4 > 2$. □

Proposition 6.2.5. $g=3 \Rightarrow v \leq 6$. *Equality can hold as can be seen by figure 6.2-3 in which $v=6$ and $g=3$.*

```
. . o .
. . . o      . o o . .      . o . o .
o . . o      o . . . o      o . . . o
o . o .      . o o . .      o . . o .
```

```
Figure 6.2-3
Some lattice polygons with g=3
and largest v
```

Proof. $v \geq 7 \Rightarrow g \geq 4 > 3$. □

Proposition 6.2.6. $g = 4 \Rightarrow v \leq 8$. *Equality holds when and only when K is lattice equivalent to the centrally symmetric octagon shown in figure 6.2-4.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 6.2-4
Unique lattice polygon with g=4
and largest v

Proof. $v \geq 9 \Rightarrow g \geq 7 > 4$. Equality is determined by the Central Octagon Theorem. □

Lemma 6.2.7. $g = 5 \Rightarrow v \leq 8$.

Proof. $v \geq 9 \Rightarrow g \geq 7 > 5$. This result is not best possible. □

Proposition 6.2.8. $g = 5 \Rightarrow v \leq 7$. *Equality can hold as can be seen by figure 6.2-5 in which $v = 7$ and $g = 5$.*

```
. o o . .        . o o . .
o . . o .        o . . . .
o . . . .        o . . . o
. o . . o        . o . o .
```

Figure 6.2-5
Some lattice polygons with g=5
and largest v

This result is due to a computer search.

This result is unusual enough to warrant calling it to the reader's attention.

Observation (The Octagon Anomaly). A lattice octagon can have 4 interior lattice points or 6 interior lattice points, but it can't have exactly 5 interior lattice points.

Proposition 6.2.9. $g = 6 \Rightarrow v \leq 8$. *Equality can hold as can be seen by figure 6.2-6 in which $v = 8$ and $g = 6$.*

```
. o o . .
o . . . o
o . . . o
. o o . .
```

Figure 6.2-6
Lattice polygon with g=6
and largest v

Proof. $v \geq 9 \Rightarrow g \geq 7 > 6$. □

Proposition 6.2.10. $g = 7 \Rightarrow v \leq 9$. *Equality occurs when and only when K is lattice equivalent to the nonagon shown in figure 6.2-7.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

Figure 6.2-7
Unique lattice polygon with g=7
and largest v

Proof. $v \geq 10 \Rightarrow g \geq 10 > 7$. (See also Proposition 6.1.23.) □

Proposition 6.2.11. $g = 8 \Rightarrow v \leq 8$. *Equality can hold as can be seen by figure 6.2-8 in which $v = 8$ and $g = 8$.*

```
. o o . . .
o . . . . o
o . . . . o
. o o . . .
```

Figure 6.2-8
Lattice polygon with g=8
and largest v

This result is due to a computer search.

Proposition 6.2.12. $g = 9 \Rightarrow v \leq 8$. *Equality can hold as can be seen by figure 6.2-9 in which $v = 8$ and $g = 9$.*

```
. o o . . . .
o . . . . . o
o . . . . o .
. o o . . . .
```

Figure 6.2-9
Lattice polygon with g=9
and largest v

This result is due to a computer search.

Proposition 6.2.13. $g = 10 \Rightarrow v \leq 10$. *Equality occurs when and only when K is lattice equivalent to the decagon shown in figure 6.2-10.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

Figure 6.2-10
Unique lattice polygon with g=10
and largest v

This result is due to a computer search.

We may summarize as follows:

Theorem 6.2.14. *Let* $v(g) = \sup\{v(K)|g(K) = g\}$. *Then*

a. $v(0) = 4$.
b. $v(1) = 6$.
c. $v(2) = 6$.
d. $v(3) = 6$.
e. $v(4) = 8$.
f. $v(5) = 7$.
g. $v(6) = 8$.
h. $v(7) = 9$.
i. $v(8) = 8$.
j. $v(9) = 8$.
k. $v(10) = 10$.

It is interesting to note that $v(g)$ is not monotone.

Section 7
Inequalities involving the number of vertices

In this section, we investigate inequalities involving v, the number of vertices of a convex lattice polygon. We note the following trivial inequalities: $b \geq v$ and $P \geq v$.

Section 7.1.
Inequalities involving number of lattice points.

In this section, we study convex lattice polygons, K, with v vertices. G denotes the number of lattice points inside or on K. We let $G(v) = \inf\{G(K)|v(K) = v\}$ and $g(v) = \inf\{g(K)|v(K) = v\}$.

Proposition 7.1.1. *If K is a convex lattice polygon with v fixed and smallest G, then K is lean.*

Proof. Suppose P is a lattice point on side A_2A_3 of polygon $A_1A_2A_3A_4A_5 \ldots A_v$. Then polygon $A_1PA_3A_4A_5 \ldots A_v$ would have the same number of vertices but smaller G. □

Corollary 7.1.2. $G(v) = g(v) + v$.

Proposition 7.1.3. *If v is fixed, then g can be arbitrarily large.*

Proof. We can expand the figure by any amount (by an integer scaling about the origin) keeping v fixed and making g get as large as we want. □

So now we know what the smallest G can be. We look at the cases of equality. The following results were established by a computer search.

Proposition 7.1.4. *If $v = 3$ then $G \geq 3$. Equality occurs when and only when K is lattice equivalent to TRIANG$(1, 1)$. See figure 7.1-1.*

```
o .
o o
```

Figure 7.1-1
Unique polygon with v=3
and smallest G

Proposition 7.1.5. *If $v = 4$ then $G \geq 4$. Equality occurs when and only when K is lattice equivalent to the unit square, TRAP$(1, 1, 1)$. See figure 7.1-2.*

```
o o
o o
```

Figure 7.1-2
Unique polygon with v=4
and smallest G

Proposition 7.1.6. *If $v = 5$ then $G \geq 6$. Equality occurs when and only when K is lattice equivalent to the pentagon shown below.*

```
. o .
o . o
o o .
```

Figure 7.1-3
Unique polygon with v=5
and smallest G

Proposition 7.1.7. *If $v = 6$ then $G \geq 7$. Equality occurs when and only when K is lattice equivalent to the centrally symmetric hexagon shown below.*

```
. o o
o . o
o o .
```

Figure 7.1-4
Unique polygon with v=6
and smallest G

Proposition 7.1.8. *If $v = 7$ then $G \geq 11$. Equality occurs when and only when K is lattice equivalent to one of the heptagons shown below.*

```
. o . .        . o o .
o . . o        . . . o
o . . o        o . . o
. o o .        o o . .
```

Figure 7.1-5
Only polygons with v=7
and smallest G

Proposition 7.1.9. *If $v = 8$ then $G \geq 12$. Equality occurs when and only when K is lattice equivalent to the centrally symmetric octagon shown below.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 7.1-6
Unique polygon with v=8
and smallest G

Proposition 7.1.10. *If $v = 9$ then $G \geq 16$. Equality occurs when and only when K is lattice equivalent to the nonagon shown below.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

Figure 7.1-7
Unique polygon with v=9
and smallest G

Proposition 7.1.11. *If $v = 10$ then $G \geq 20$. Equality occurs when and only when K is lattice equivalent to the decagon shown below.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

Figure 7.1-8
Unique Polygon with v=10
and smallest G

We may summarize this data as follows:

Theorem 7.1.12. *Let $G(v) = \inf\{G(K)|v(K) = v\}$. Then*

a. $G(3) = 3$.
b. $G(4) = 4$.
c. $G(5) = 6$.
d. $G(6) = 7$.
e. $G(7) = 11$.
f. $G(8) = 12$.
g. $G(9) = 16$.
h. $G(10) = 20$.
i. $G(v) = g(v) + v$.

Section 7.2.
Inequalities involving the Area.

Proposition 7.2.1. *If v is fixed, then A can be arbitrarily large.*

Proof. We can expand the figure by any amount (by an integer scaling about the origin) keeping v fixed and making A get larger. □

Proposition 7.2.2. *If K is a convex lattice polygon with v fixed and smallest A, then K is lean.*

Proof. Suppose P is a lattice point on side A_2A_3 of polygon $A_1A_2A_3A_4A_5\ldots A_v$. Then polygon $A_1PA_3A_4A_5\ldots A_v$ would have the same number of vertices but smaller A. □

Proposition 7.2.3. $g > 1 \Rightarrow A \geq v - 4$.

Proof. $g > 1 \Rightarrow b \leq A + 4$ (Proposition 5.1.5) and $v \leq b \Rightarrow A \geq b - 4 \geq v - 4$. □

Proposition 7.2.4. $A \geq (v - 2)/2$.

Proof. Starting from Pick's Formula, we have $A = b/2 + g - 1 \geq b/2 - 1 \geq v/2 - 1$. □

Proposition 7.2.5. $A(v) = G(v) - v/2 - 1$.

This follows immediately from Pick's formula, $A = G - b/2 - 1$ and Proposition 7.2.2.

Furthermore, the figures for which equality holds are exactly the same figures as were exhibited in section 7.1 for the cases when equality for G holds.

We may summarize this data as follows:

Theorem 7.2.6. *Let $A(v) = \min\{A(K)|v(K) = v\}$. Then*

a. $A(3) = 1/2$.
b. $A(4) = 1$.
c. $A(5) = 5/2$.
d. $A(6) = 3$.
e. $A(7) = 13/2$.
f. $A(8) = 7$.
g. $A(9) = 21/2$.
h. $A(10) = 14$.
i. $A(v) = G(v) - v/2 - 1$.

Section 7.3.
Inequalities involving the Diameter.

Let D denote the diameter of a convex lattice polygon, K, with v vertices.

We investigated by computer the relationship between D and v for all convex lattice polygons with $D \leq 10$ and $g \leq 10$. In each case, we found the minimum value of D, which was always less than 6.

Proposition 7.3.1 (effectiveness of search for minimum D). *No lattice polygons with $D \leq 4$ were missed by the computer search.*

Proof. Consider any lattice polygon with $v \leq 10$ and $D \leq 4$. Since $g \leq (D-1)^2$ (by Proposition 5.6.15), we have $g \leq (4-1)^2 = 9$ and so this polygon was found by the search since we examined all polygons with $D \leq 10$ and $g \leq 10$. □

This proves that the first 6 of the following 8 propositions is true. The last two, with $D \geq 5$ are only conjectures.

The following results were found by computer:

Proposition 7.3.2. *If $v = 3$ then $D \geq \sqrt{2}$. Equality occurs when and only when K is lattice congruent to TRIANG$(1,1)$. See figure 7.3-1.*

```
o .
o o
```

```
Figure 7.3-1
Unique polygon with v=3
and smallest diameter
```

Proposition 7.3.3. *If $v = 4$ then $D \geq \sqrt{2}$. Equality occurs when and only when K is lattice congruent to the unit square, TRAP$(1,1,1)$. See figure 7.3-2.*

```
o o
o o
```

```
Figure 7.3-2
Unique polygon with v=4
and smallest diameter
```

Proposition 7.3.4. *If $v = 5$ then $D \geq \sqrt{5}$. Equality occurs when and only when K is lattice congruent to one of the pentagons pictured below.*

```
. o .        . o .
o . o        o . o
o . o        o o .
```

Figure 7.3-3
Only polygons with v=5
and smallest diameter

Proposition 7.3.5. *If $v = 6$ then $D \geq 2\sqrt{2}$. Equality occurs when and only when K is lattice congruent to the centrally symmetric hexagon shown below.*

```
. o o
o . o
o o .
```

Figure 7.3-4
Unique polygon with v=6
and smallest diameter

Proposition 7.3.6. *If $v = 7$ then $D \geq \sqrt{10}$. Equality occurs when and only when K is lattice congruent to the heptagon shown below.*

```
. o . .
o . . o
o . . o
. o o .
```

Figure 7.3-5
Unique polygon with v=7
and smallest diameter

Proposition 7.3.7. *If $v = 8$ then $D \geq \sqrt{10}$. Equality occurs when and only when K is lattice congruent to the centrally symmetric octagon shown below.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 7.3-6
Unique polygon with v=8
and smallest diameter

Proposition 7.3.8. *If $v = 9$ then $D \geq 5$. Equality occurs when and only when K is lattice congruent to the nonagon shown below.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

Figure 7.3-7
Unique polygon with v=9
and smallest diameter

Proposition 7.3.9. *If $v = 10$ then $D \geq \sqrt{29}$. Equality occurs when and only when K is lattice congruent to the decagon shown below.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

Figure 7.3-8
Unique Polygon with v=10
and smallest diameter

We may summarize this data as follows:

Theorem 7.3.10. *Let $D(v) = \min\{D(K)|v(K) = v\}$. Then*
a. $D(3) = \sqrt{2}$.
b. $D(4) = \sqrt{2}$.
c. $D(5) = \sqrt{5}$.
d. $D(6) = 2\sqrt{2}$.
e. $D(7) = \sqrt{10}$.
f. $D(8) = \sqrt{10}$.
g. $D(9) = 5$.
h. $D(10) = \sqrt{29}$.

Section 7.4.
Inequalities involving the Perimeter.

In this section, we study convex lattice polygons with v vertices and perimeter P.

Proposition 7.4.1. *For all convex lattice polygons, $P \geq 4 + (v-4)\sqrt{2}$.*

Proof. This formula is easily checked for $v = 3$. If $v \geq 4$, then at most 4 vertices can have the smallest possible length of 1. The remaining $v - 4$ vertices must have length at least the next largest value of $\sqrt{2}$. □

Proposition 7.4.2. *If K is a convex lattice polygon with v fixed and smallest P, then K is lean.*

Proof. Suppose Q is a lattice point on side A_2A_3 of polygon $A_1A_2A_3A_4A_5 \ldots A_v$. Then polygon $A_1QA_3A_4A_5 \ldots A_v$ would have the same number of vertices but smaller P. □

Proposition 7.4.3. *If $v = 3$ then $P \geq 2 + \sqrt{2} \approx 3.414$. Equality occurs when and only when K is lattice congruent to TRIANG$(1,1)$. See figure 7.4-1.*

```
o .
o o
```

```
    Figure 7.4-1
Unique polygon with v=3
   and smallest P
```

Proposition 7.4.4. *If $v = 4$ then $P \geq 4$. Equality occurs when and only when K is lattice congruent to the unit square, TRAP$(1,1,1)$. See figure 7.4-2.*

```
o o
o o
```

```
    Figure 7.4-2
Unique polygon with v=4
   and smallest P
```

Proposition 7.4.5. *If $v = 5$ then $P \geq 2 + 3\sqrt{2} \approx 6.243$. Equality occurs when and only when K is lattice congruent to the pentagon shown below.*

```
. o .
o . o
o o .
```

Figure 7.4-3
Unique polygon with v=5
and smallest P

Proposition 7.4.6. *If $v = 6$ then $P \geq 4 + 2\sqrt{2} \approx 6.828$. Equality occurs when and only when K is lattice congruent to the centrally symmetric hexagon shown below.*

```
. o o
o . o
o o .
```

Figure 7.4-4
Unique polygon with v=6
and smallest P

Proposition 7.4.7. *If $v = 7$ then $P \geq 3 + 3\sqrt{2} + \sqrt{5} \approx 9.479$. Equality occurs when and only when K is lattice congruent to the heptagon shown below.*

```
. o . .
o . . o
o . . o
. o o .
```

Figure 7.4-5
Unique polygon with v=7
and smallest P

Proposition 7.4.8. *If $v = 8$ then $P \geq 4+4\sqrt{2} \approx 9.657$. Equality occurs when and only when K is lattice congruent to the centrally symmetric octagon shown below.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 7.4-6
Unique polygon with v=8
and smallest P

Proposition 7.4.9. *If $v = 9$ then $P \geq 4 + 3\sqrt{2} + 2\sqrt{5} \approx 12.715$. Equality occurs when and only when K is lattice congruent to the nonagon shown below.*

```
. . o o .
. o . . o
. . . . o
o . . o .
o o . . .
```

```
Figure 7.4-7
Unique polygon with v=9
and smallest P
```

Proposition 7.4.10. *If* $v = 10$ *then* $P \geq 4+4\sqrt{2}+2\sqrt{5} \approx 14.129$. *Equality occurs when and only when* K *is lattice congruent to the decagon shown below.*

```
. . . o o .
. o . . . o
o . . . . o
o . . . o .
. o o . . .
```

```
Figure 7.4-8
Unique Polygon with v=10
and smallest P
```

We may summarize this data as follows:

Theorem 7.4.11. *Let* $P(v) = \inf\{P(K)|v(K) = v\}$. *Then*

a. $P(3) = 2 + \sqrt{2}$.
b. $P(4) = 4$.
c. $P(5) = 2 + 3\sqrt{2}$.
d. $P(6) = 4 + 2\sqrt{2}$.
e. $P(7) = 3 + 3\sqrt{2} + \sqrt{5}$.
f. $P(8) = 4 + 4\sqrt{2}$.
g. $P(9) = 4 + 3\sqrt{2} + 2\sqrt{5}$.
h. $P(10) = 4 + 4\sqrt{2} + 2\sqrt{5}$.

Section 7.5.
Inequalities involving boundary lattice points.

Proposition 7.5.1. *Let* $b(v) = \inf\{v(K)|v(K) = v\}$. *Then* $b(v) = v$ *for all* v.

Proof. This follows from the fact that we have previously exhibited a lean polygon with exactly v vertices for any v (Proposition 6.1.30). $\square$

Proposition 7.5.2. *If* v *is fixed, then* b *can be arbitrarily large.*

Proof. We can expand the figure by applying a similarity transformation about the origin with ratio of similitude being any positive integer. This transformation keeps v fixed and allows us to make b as large as we want. $\square$

Section 8
Miscellaneous inequalities

In this section, we investiate some miscellaneous inequalities.

Section 8.1.
Inequalities involving b.

Proposition 8.1.1. *If $g > 1$, then $A \geq b - 4$.*

Proof. From Scott's Bound for b, we have $b \leq 2g + 6$. Combining this with Pick's Formula, we get $A = b/2 + g - 1 \geq b/2 + (b-6)/2 - 1 = b - 4$. □

Proposition 8.1.2. *If $g > 0$, then $b \leq A + 7/2$.*

Proof. Similarly, from $b \leq 2g + 7$ we get $2A = b + 2g - 2$, so $2A \geq 2b - 7$ or $2b \leq 2A + 7$ which is equivalent to our result. □

Proposition 8.1.3. *$v \leq 2A + 2$.*

This follows from Proposition 7.2.4 or Proposition 8.2.4.

Proposition 8.1.4. *A can get arbitrarily large for a fixed b.*

Proof. Pick $b - 1$ consecutive points on the positive x-axis beginning with the origin. Let (x, y) be the final vertex of a triangle. $A = (0, 0)$, $B = (b - 2, 0)$. We want $\gcd(x, y) = 1$ and $\gcd(x - b + 2, y) = 1$, so let $x = 1$. Then we need $\gcd(b - 3, y) = 1$. Let $y = k(b - 3) + 1$. In other words, $C = (1, k(b - 3))$. As k increases, so does y and hence also so does A. □

Proposition 8.1.5. *If $g > 0$ then $3b \leq 2G + 7$. Equality holds when and only when K is lattice equivalent to TRIANG$(3, 3)$.*

Proof. Starting with Scott's bound for b, we have $b \leq 2g + 7 = 2(G - b) + 7$ from which the result follows. □

Proposition 8.1.6. *If G is fixed, then b can get as large as G and as small as 3.*

Proof. Since $G = b + g$, we clearly have $b \leq G$ with equality when and only when $g = 0$. Furthermore, b can get as small as 3 since we have already seen that we can have lean triangles with any value of g. □

The polygons in which equality holds for G versus b inequalities are the same polygons for which equality holds in g versus b inequalities.

For a given b, it is easy to compute the minimum P. Just use a rectangle of height 1 or TRAP(p, q) with $p = q + 1$. If we let $P(b) = \inf P(K)|b(K) = b$, then we easily see that $P(2n) = 2n$ and $P(2n + 1) = 2n + \sqrt{2}$. In other words, we have proven the following.

Proposition 8.1.7. *If $P(b) = \inf\{P(K)|b(K) = b\}$, then*

$$P(b) = b + \frac{1 - (-1)^b}{2}(\sqrt{2} - 1).$$

Section 8.2.
Inequalities involving Area.

In this section we study convex lattice polygons with area A.

Section 8.2a.
Inequalities involving A and g.

Proposition 8.2.1. *If K is a convex polygon, then $g \le A - 1/2$. Equality occurs when and only when K is a lean triangle of non-integral area. Also, $g \le \lfloor A - 1/2 \rfloor$. In this case, equality occurs when and only when K is a lean triangle or a lean quadrilateral or a triangle with $b = 4$.*

This follows from Proposition 5.2.1 and Pick's Formula.

Proposition 8.2.2. *If A is fixed, g can get as small as 0.*

Proof. TRAP$(p, q, 1)$ has $g = 0$ and can be made to have any area that is a multiple of 1/2 by suitably picking p and q. □

Section 8.2b.
Inequalities involving A and b.

Proposition 8.2.3. *If $2A$ is odd, then $b \ge 3$ with equality when and only when K is a lean triangle of area A. If $2A$ is even, then $b \ge 4$ with equality when and only when K is a lean triangle of area A.*

In other words, $b \ge (7 + (-1)^{2A})/2$.

Proof. Clearly $b \ge 3$. If $2A$ is even, then from Pick's Formula we have $b = 2A + 2 - 2g$, so we see that b must be even, or $b \ge 4$.

When $2A$ is odd, equality holds when and only when $A = g + 1/2$; i.e., when and only when K is a lean triangle of area A.

When $2A$ is even, equality holds when and only when $A = g + 1$. □

Proposition 8.2.4. *$b \le 2A + 2$. Equality holds when and only when $g = 0$.*

Proof. From Pick's Formula we have $b = 2A - 2g + 2 \le 2A + 2$ with equality when and only when $g = 0$. □

Section 8.2c.
Inequalities involving A and G.

Proposition 8.2.5. *$G \le 3A + 3/2$. Equality holds when and only when K is lattice equivalent to TRIANG(1,1).*

This follows from propositions 8.2.4 and 8.2.1.

Proposition 8.2.6. *$G \ge \lfloor A \rfloor + 3$. Equality holds when and only when K is a lean triangle or a lean quadrilateral or a triangle with $b = 4$.*

Proof. $G = b + g = 2A - g + 2$. Thus, for fixed A, G is minimized when g is maximized, that is, when $g = \lfloor A - 1/2 \rfloor$. In that case, $G \ge 2A - \lfloor A + 1/2 \rfloor + 3 = \lceil A - 1/2 \rceil + 3 = \lfloor A \rfloor + 3$. □

Proposition 8.2.7. *If $g > 0$ then $4G \le 6A + 13$. Equality holds when and only when K is lattice equivalent to TRIANG$(3, 3)$.*

Proof. From Pick's Formula, we have $b = 2G - 2A - 2$. Substituting this in Proposition 8.1.5 gives $6G - 6A - 6 = 3b \le 2G + 7$, so $4G \le 6A + 13$. □

Proposition 8.2.8. *For any lattice polygon, K, $G \geq A + 5/2$. Equality holds when and only when K is a lean triangle.*

Proof. Combining Proposition 1.2.2 with Proposition 5.2.1 yields

$$G = 2A - g + 2 \geq 2A - (A - \frac{1}{2}) + 2 = A + \frac{5}{2}$$

as claimed. □

Section 8.2d.
Inequalities involving A and v.

Proposition 8.2.9. *For A fixed, $v \geq 3$. Equality holds when and only when K is a lean triangle of area A.*

This is an immediate consequence of Proposition 3.5.

Section 8.3.
Inequalities involving A and P.

In this section, we study some isoperimetric inequalities involving A and P, the area and perimeter of a convex lattice polygon, respectively. Instead of fixing P and asking for the largest A, we will fix A and ask for the smallest P.

Proposition 8.3.1. *If $A = 1/2$ then $P \geq 2 + \sqrt{2} \approx 3.414$. Equality occurs when and only when K is lattice congruent to TRIANG$(1,1)$. See figure 8.3-1.*

```
o .
o o
```

Figure 8.3-1
Unique polygon with A=1/2
and smallest P

Proposition 8.3.2. *If $A = 1$ then $P \geq 4$. Equality occurs when and only when K is lattice congruent to the unit square, TRAP$(1,1,1)$. See figure 8.3-2.*

```
o o
o o
```

Figure 8.3-2
Unique polygon with A=1
and smallest P

Proposition 8.3.3. *If $A = 3/2$ then $P \geq 4 + \sqrt{2} \approx 5.414$. Equality occurs when and only when K is lattice congruent to TRAP$(2,1,1)$.*

```
o o .
o . o
```

Figure 8.3-3
Unique polygon with A=3/2
and smallest P

Proposition 8.3.4. *If $A = 2$ then $P \geq 4\sqrt{2} \approx 5.657$. Equality occurs when and only when K is lattice congruent to the diamond shown below.*

```
. o .
o . o
. o .
```

Figure 8.3-4
Unique polygon with A=2
and smallest P

Proposition 8.3.5. *If $A = 5/2$ then $P \geq 2 + 3\sqrt{2} \approx 6.243$. Equality occurs when and only when K is lattice congruent to the pentagon shown below.*

```
. o .
o . o
o o .
```

Figure 8.3-5
Unique polygon with A=5/2
and smallest P

Proposition 8.3.6. *If $A = 3$ then $P \geq 4 + 2\sqrt{2} \approx 6.828$. Equality occurs when and only when K is lattice congruent to one of the polygons shown below.*

```
. o .      . o o
o . o      o . o
o . o      o o .
```

Figure 8.3-6
Only polygons with A=3
and smallest P

Proposition 8.3.7. *If $A = 7/2$ then $P \geq 6 + \sqrt{2} \approx 7.414$. Equality occurs when and only when K is lattice congruent to the pentagon shown below.*

```
o o .
. . o
o . o
```

Figure 8.3-7
Unique polygon with A=7/2
and smallest P

Proposition 8.3.8. *If $A = 4$ then $P \geq 2 + 4\sqrt{2} \approx 7.657$. Equality occurs when and only when K is lattice congruent to the hexagon shown below.*

```
. o o .
o . . o
. o o .
```

Figure 8.3-8
Unique polygon with A=4
and smallest P

Proposition 8.3.9. *If $A = 9/2$ then $P \geq 4 + 3\sqrt{2} \approx 8.253$. Equality occurs when and only when K is lattice congruent to the hexagon shown below.*

```
. o o .
o . . o
o . o .
```

```
Figure 8.3-9
Unique polygon with A=9/2
and smallest P
```

Proposition 8.3.10. *If $A = 5$ then $P \geq 6 + 2\sqrt{2} \approx 8.828$. Equality occurs when and only when K is lattice congruent to one of the polygons shown below.*

```
. o . o     . o o .     o . o .
o . . o     o . . o     . . . o
o . o .     o . . o     o . o .
```

```
Figure 8.3-10
Only polygons with A=5
and smallest P
```

Proposition 8.3.11. *If $A = 11/2$ then $P \geq 2 + 5\sqrt{2} \approx 9.071$. Equality occurs when and only when K is lattice congruent to the heptagon shown below.*

```
. o . .
o . . .
o . . o
. o o .
```

```
Figure 8.3-11
Unique polygon with A=11/2
and smallest P
```

Proposition 8.3.12. *If $A = 6$ then $P \geq 2 + 2\sqrt{2} + 2\sqrt{5} \approx 9.301$. Equality occurs when and only when K is lattice congruent to one of the polygons shown below. These polygons may be obtained from the centrally symmetric octagon by removing any two non-consecutive vertices.*

```
. . o .     . . o .     . . o .     . . o .
. . . o     o . . o     o . . o     o . . o
o . . o     . . . o     o . . o     o . . o
. o o .     . o o .     . . o .     . o . .
```

```
Figure 8.3-12
Only polygons with A=6
and smallest P
```

Proposition 8.3.13. *If $A = 13/2$ then $P \geq 3 + 3\sqrt{2} + \sqrt{5} \approx 9.479$. Equality occurs when and only when K is lattice congruent to the heptagon shown below.*

```
. o . .
o . . o
o . . o
. o o .
```

```
        Figure 8.3-13
 Unique polygon with A=13/2
      and smallest P
```

Proposition 8.3.14. *If $A = 7$ then $P \geq 4 + 4\sqrt{2} \approx 9.657$. Equality occurs when and only when K is lattice congruent to the octagon shown below.*

```
. o o .
o . . o
o . . o
. o o .
```

```
        Figure 8.3-14
  Unique polygon with A=7
      and smallest P
```

Proposition 8.3.15. *If $A = 15/2$ then $P \geq 6 + 3\sqrt{2} \approx 10.243$. Equality occurs when and only when K is lattice congruent to the heptagon shown below.*

```
. o o .
o . . o
. . . o
o . o .
```

```
        Figure 8.3-15
 Unique polygon with A=15/2
      and smallest P
```

Proposition 8.3.16. *If $A = 8$ then $P \geq 8 + 2\sqrt{2} \approx 10.828$. Equality occurs when and only when K is lattice congruent to one of the hexagons shown below.*

```
. o o .        . o . o
o . . o        o . . .
. . . .        . . . o
o . . o        o . o .
```

```
        Figure 8.3-16
  Only polygons with A=8
      and smallest P
```

Proposition 8.3.17. *If $A = 17/2$ then $P \geq 4 + 5\sqrt{2} \approx 11.071$. Equality occurs when and only when K is lattice congruent to the heptagon shown below.*

```
. o . o .
o . . . o
. . . . o
. . o o .
```

Figure 8.3-17
Unique polygon with A=17/2
and smallest P

We may summarize this data as follows:

Theorem 8.3.18. *Let* $P(A) = \inf\{P(K)|A(K) = A\}$*. Then*

a. $A(1/2) = 2 + \sqrt{2}$.
b. $A(1) = 4$.
c. $A(3/2) = 4 + \sqrt{2}$.
d. $A(2) = 4\sqrt{2}$.
e. $A(5/2) = 2 + 3\sqrt{2}$.
f. $A(3) = 4 + 2\sqrt{2}$.
g. $A(7/2) = 6 + \sqrt{2}$.
h. $A(4) = 2 + 4\sqrt{2}$.
i. $A(9/2) = 4 + 3\sqrt{2}$.
j. $A(5) = 6 + 2\sqrt{2}$.
k. $A(11/2) = 2 + 5\sqrt{2}$.
l. $A(6) = 2 + 2\sqrt{2} + 2\sqrt{5}$.
m. $A(13/2) = 3 + 3\sqrt{2} + \sqrt{5}$.
n. $A(7) = 4 + 4\sqrt{2}$.
o. $A(15/2) = 6 + 3\sqrt{2}$.
p. $A(8) = 8 + 2\sqrt{2}$.
q. $A(17/2) = 4 + 5\sqrt{2}$.

Section 8.4.
Miscellaneous.

Problem. Can we find a formula for the area of a triangle in terms of the lattice lengths of the sides?

Answer. No, the area depends on g. Even if we are given g, the triangle is not determined.

```
. . o          . . . o
. . .          . . . .
. . .          . . + .
. + .          . . . .
. + .          . + . .
o . o          o o . .
```

Figure 8.4-1
Triangle is not determined by lattice lengths
of sides and g

```
. . o          . . . o
. . .          . . . .
. . .          . . . .
. . .          . . + .
. + .          . . . .
. + .          . + . .
. + .          . + . .
o . o          o o . .
```

Figure 8.4-2
Totally inequivalent triangles with sides
of same lattice length and same g

Section 8.5.
Final Summary.

We summarize the inequalities found so far below:

Definition. If $\mathbf{F}$ and $\mathbf{f}$ are any two functions with domain the set of convex lattice polygons, we define

$$\mathbf{F}^+(f) = \sup\{\mathbf{F}(K) | \mathbf{f}(K) = f\}$$

$$\mathbf{F}^-(f) = \inf\{\mathbf{F}(K) | \mathbf{f}(K) = f\}.$$

Note that the free variable, f, is a variable whose name happens to be the same as the name of the function $\mathbf{f}$.

In plain English, $F^+(f)$ is the largest that F can get when f is fixed. Similarly, $F^-(f)$ is the smallest that F can get when f is fixed.

Theorem 8.5.1.

a.
$$b^+(g) = \begin{cases} \infty, & \text{if } g = 0 \\ 9, & \text{if } g = 1 \\ 2g+6, & \text{if } g > 1 \end{cases}$$

b.
$$b^-(g) = 3$$

c.
$$A^+(g) = \begin{cases} \infty, & \text{if } g = 0 \\ 9/2, & \text{if } g = 1 \\ 2g+2, & \text{if } g > 1 \end{cases}$$

d.
$$A^-(g) = g + \frac{1}{2}$$

e.
$$P^+(g) = \infty$$

f.
$$2(\lceil\sqrt{g}\rceil + 1) \leq P^-(g) > \sqrt{12g+6}$$

g.
$$D^+(g) = \infty$$

h.
$$\lceil\sqrt{g}\rceil + 1 \leq D^-(g) \leq (\lceil\sqrt{g}\rceil + 1)\sqrt{2}$$

i.
$$\lceil\sqrt{g}\rceil + 1 \leq w^+(g) \leq \sqrt{2(g+1)\sqrt{3}}$$

j.
$$w^-(g) = 0$$

k.
$$v^-(g) = 3$$

l.
$$g^+(v) = \infty$$

m. $$3\lfloor\frac{v-1}{2}\rfloor - 5 \leq g^{-}(v) \leq \binom{\lceil v/2\rceil}{3}$$

n. $$G^{+}(v) = \infty$$

o. $$G^{-}(v) = g^{-}(v) + v$$

p. $$A^{+}(v) = \infty$$

q. $$A^{-}(v) = g^{-}(v) + \frac{v}{2} - 1$$

r. $$b^{+}(v) = \infty$$

s. $$b^{-}(v) = v$$

t. $$g^{+}(A) = \lfloor A - \frac{1}{2}\rfloor$$

u. $$g^{-}(A) = 0$$

v. $$b^{+}(A) = 2A + 2$$

w. $$b^{-}(A) = \frac{(7 + (-1)^{2A})}{2}$$

x. $$G^{+}(A) = 3A + \frac{3}{2}$$

y. $$G^{-}(A) = \lfloor A\rfloor + 3$$

z. $$v^{-}(A) = 3$$

aa. $$b^{+}(G) = G$$

bb. $$b^{-}(G) = 3$$

cc. $$A^{+}(b) = \begin{cases} \infty, & \text{if } g = 0 \\ b - 7/2, & \text{if } g = 1 \\ b - 4, & \text{if } g > 1 \end{cases}$$

dd. $$P^-(b) = b + \frac{1-(-1)^b}{2}(\sqrt{2}-1).$$

Proof.

a. Proposition 5.1.2.
b. Proposition 5.1.1.
c. Proposition 5.2.2.
d. Proposition 5.2.1.
e. Proposition 5.5.1.
f. Theorem 5.5.18m.
g. Proposition 5.6.1.
h. Theorem 5.6.22x.
i. Theorem 5.7.17l.
j. Proposition 5.7.2.
k. Proposition 3.4.
l. Section 6.1.
m. Theorem 6.1.32q.
n. Obvious.
o. Theorem 7.1.12i.
p. Proposition 7.2.1.
q. Theorem 7.2.6i.
r. Proposition 7.5.2.
s. Proposition 7.5.1.
t. Proposition 8.2.1.
u. Proposition 8.2.2.
v. Proposition 8.2.4.
w. Proposition 8.2.3.
x. Proposition 8.2.5.
y. Proposition 8.2.6.
z. Proposition 8.2.9.
aa. Proposition 8.1.6.
bb. Proposition 8.1.6.
cc. Propositions 8.1.1, 8.1.2, and 8.1.4.
dd. Proposition 8.1.7. $\square$

Section 9
Characterization of convex lattice polygons with small g

In 1980, Arkinstall [3] proved that (up to lattice equivalence) there is just one lattice hexagon containing a single interior lattice point. In this section, we are interested in finding all lattice polygons with at most one interior lattice point. As usual, g denotes the number of lattice points in the interior of the convex lattice polygon being considered.

Section 9.1.
Characterization of convex lattice polygons with g=0.

Theorem 9.1.1 (Characterization of convex lattice polygons with $g = 0$). *If K is a lattice polygon with $g = 0$, then K is lattice equivalent to one of the following polygons:*

1. *TRIANG$(p, 1)$ where p is any positive integer.*
2. *TRIANG$(2, 2)$.*
3. *TRAP$(p, q, 1)$ where p and q are any positive integers.*

Proof. K must have fewer than 5 sides because the Lattice Pentagon Theorem shows that if K had 5 or more sides, it would contain a lattice point. We thus need only consider two cases: triangles and quadrilaterals.
Case 1: The polygon is a triangle.

Using the x-axis Lemma, we can find an integral unimodular affine transformation that maps the largest side into the positive x-axis, with one vertex, A, at the origin, a second vertex, B, on the x-axis at $(p, 0)$ where p is a positive integer. The third vertex, C, maps to some point above the x-axis. Let h denote the height of vertex C from the x-axis.

If $h = 1$, then C is at height 1 above the x-axis, so we can apply a shear about the x-axis to move point C to the y-axis. This shows that in this case, the polygon is equivalent to TRIANG$(p, 1)$.

So assume that $h > 1$. Let r be the length of the line segment formed by the intersection of the line $y = 1$ with triangle ABC. (The relative interior of this segment lies wholly within $\triangle ABC$ because $h > 1$.) Clearly $p > r$.

By considering similar triangles we get $r/(h-1) = p/h$. Solving for r gives

$$r = p\frac{h-1}{h}.$$

Since we must have $r \leq 1$ (otherwise the segment will contain some lattice point in its interior), we find that $p(h-1)/h \leq 1$ from which we can conclude that either $p = 1$ or

$$h \leq 1 + \frac{1}{p-1}. \qquad (*)$$

If $p = 1$, then h can be arbitrary, and we find that the triangle is equivalent to TRIANG$(h, 1)$.

If $p > 1$, then from (*) and the fact that $h > 1$, we see that $p = 2$ and $h \leq 2$. So $h = 2$ and $p = 2$.

Thus $r = p(h-1)/h = 1$. If the line $y = 1$ meets AC at E, then E is at height 1 above the x-axis, so we can find a shear that will move E onto the y-axis. This shows that the triangle is equivalent to TRIANG$(2, 2)$.

Case 2: The polygon is a quadrilateral.

Polygon K must be a trapezoid, for a non-trapezoidal quadrilateral must contain a lattice point by the Lattice Trapezium Theorem.

Using the x-axis Lemma, we can find an integral unimodular affine transformation that maps the larger base into the positive x-axis, with one vertex, A, at the origin, a second vertex, B, on the x-axis at $(p, 0)$ where p is a positive integer, and the other two vertices, C and D, being above the x-axis. Since unimodular affine transformations preserve parallelism, we have $CD \parallel AB$. Label the vertices so that D is to the left of C. Ratios on parallel line segments are preserved, so $AB \geq DC$. Let q be the length of DC and let h be the height of the trapezoid, that is, the distance between AB and DC. We have $p \geq q > 0$.

If $h = 1$, then D is at height 1 above the x-axis, so we can apply a shear about the x-axis to move point D to the y-axis. This shows that in this case, the polygon is equivalent to TRAP$(p, q, 1)$.

So now assume that $h > 1$. Consider the line $y = 1$. This line meets AD at E and BC at F. Since $h > 1$, the interior of segment EF lies within the interior of trapezoid $ABCD$. Draw in diagonal BD and let BD meet EF at G. Let r be the length of the line segment EF.

By considering similar triangles, we get

$$r = \frac{h-1}{h}p + \frac{1}{h}q.$$

We will now show that $p = 1$. For suppose $p > 1$. Then since $q \geq 1$, we would have

$$r = \frac{h-1}{h}p + \frac{1}{h}q > \frac{h-1}{h} + \frac{1}{h} = 1.$$

However, r can not be greater than 1; if r were greater than 1, segment EF would have to contain a lattice point in its interior. This would be a contradiction for it would produce a lattice point inside quadrilateral $ABCD$. Hence $p = 1$.

Since $q \leq p$, we must have $q = 1$. Thus $ABCD$ is a parallelogram with bases both of length 1. If each of the other two sides have lattice length t, then this parallelogram is equivalent to TRAP$(t, t, 1)$ and we are done. $\square$

Section 9.2.
Characterization of polygons with g=1.

In 1980, Arkinstall characterized those lattice hexagons for which $g = 1$. (See Arkinstall [3]). In this section, we will complete the study by characterizing all lattice polygons with $g = 1$.

Proposition 9.2.1 (Characterization of lattice triangles with $g = 1$). *If K is a lattice triangle with $g = 1$, then K is lattice equivalent to one of the following five triangles:*

```
o . . . .     o . . .      . o .
. + . . .     . + . .      . + .
o . . . o     o . . o      o . o

   (a)           (b)          (c)


o . . .
. . . .       . . o
. + . .       o + .
o . . o       . o .

  (d)          (e)
```

```
    Figure 9.2-1
All lattice triangles with g=1
```

Proof. Using the x-axis Lemma, we can apply an integral unimodular affine transformation to map the side of the triangle with largest lattice length onto the x-axis, with A at the origin and B at $(p, 0)$. The third point, C, maps into a point above the x-axis. Let h be the height of the triangle, that is, the distance from C to the x-axis. We find that h must be larger than 1 otherwise the triangle will have no interior lattice points.

Let r be the length of the segment intercepted by the triangle on the line $y = 1$. Since $h > 1$, this segment goes through the interior of the triangle, and we must have $r \leq 2$ otherwise this segment would contain 2 lattice points in its interior. By similar triangles, we see that $(h - 1)/r = h/p$.

But $r \leq 2$ implies that

$$p = \frac{hr}{h-1} \leq \frac{2h}{h-1} \leq 4.$$

Case 1: $h = 2$:

In this case, $p = 2r \leq 4$, so $p = 1, 2, 3$, or 4. Point C lies along $y = 2$ and there at most two locations for C to get inequivalent triangles, namely, $C = (j, 2), j = 0, 1$.

Case 1a: $h = 2$, $p = 4$:

Choice $C = (0, 2)$ produces TRIANG$(4, 2)$ with $g = 1$. Choice $C = (1, 2)$ is ruled out because $\triangle ABC$ has $g = 2$. This triangle is shown in figure 9.2-1a.

Case 1b: $h = 2$, $p = 3$:

Both $C = (0, 2)$ and $C = (1, 2)$ yield triangles with $g = 1$. However, we need only include one of these, namely, TRIANG$(3, 2)$, because the other one is equivalent to this one after a reflection about $x = 3/2$ followed by a unit shear about the x-axis. This triangle is shown in figure 9.2-1b.

Case 1c: $h = 2$, $p = 2$:

Choice $C = (0, 2)$ is ruled out because the resulting triangle has no interior lattice points. Choice $C = (1, 2)$ yields a valid triangle and is included in our classification. This triangle is shown in figure 9.2-1c.

Case 1d: $h = 2$, $p = 1$:

Both $C = (0, 2)$ and $C = (1, 2)$ are ruled out because the resulting triangles have no interior lattice points.

Case 2: $h = 3$:

In this case, $p = 3r/2 \leq 3$, so $p = 1, 2$, or 3. Point C lies along $y = 3$ and there at most three locations for C to get inequivalent triangles, namely, $C = (j, 3), j = 0, 1, 2$.

Case 2a: $h = 3$, $p = 3$:

Both $C = (1, 3)$ and $C = (2, 3)$ give rise to triangles with more than 1 interior lattice point. $C = (0, 3)$ gives rise to a valid triangle, namely, TRIANG$(3, 3)$. This triangle is shown in figure 9.2-1d.

Case 2b: $h = 3$, $p = 2$:

Choice $C = (1, 3)$ yields a triangle with 2 interior lattice points. Choices $C = (0, 3)$ and $C = (2, 3)$ are valid and equivalent, but both have a side of lattice length 3 and are equivalent to TRIANG$(3, 2)$ discovered in case 1b.

Case 2c: $h = 3$, $p = 1$:

Choices $C = (0, 3)$ and $C = (1, 3)$ yield triangles with no interior lattice points and are ruled out. Choice $C = (2, 3)$ generates a valid triangle. This case is shown in the figure below.

```
. . o
. . .
. + .
o o .
```

A unit shear shows this triangle to be equivalent to the one in figure 9.2-1e.

Case 3: $h = 4$:

In this case, $p = 4r/3 \leq 8/3$, so $p = 1$ or 2. Point C lies along $y = 4$ and there at most four locations for C to get inequivalent triangles, namely, $C = (j, 4), j = 0, 1, 2, 3$.

Case 3a: $h = 4$, $p = 2$:

Choices $C = (0, 4)$ and $C = (2, 4)$ are valid and both equivalent to TRAP$(4, 2)$ already covered by case 1a. Choices $C = (3, 4)$ and $C = (1, 4)$ are ruled out because they yield triangles with too many interior lattice points.

Case 3b: $h = 4$, $p = 1$:

Choices $C = (0, 4)$ and $C = (1, 4)$ are ruled out because they generate triangles with no interior lattice points. Choices $C = (2, 4)$ and $C = (3, 4)$ yield valid triangles, but they have been included already in this classification because one of their sides has lattice length 2 and AB has lattice length 1 so is not the side of shortest lattice length in the triangle.

Case 4: $h \geq 5$:

In this case, since $r \leq 2$,

$$p = r\frac{h}{h-1} \leq \frac{2h}{h-1} \leq \frac{2h}{4} \leq \frac{h}{2} \leq \frac{5}{2},$$

so $p = 1$ or 2. Point C lies along $y = h$ and there at most h locations for C to get inequivalent triangles, namely, $C = (j, h)$, $j = 0, 1, 2 \ldots, h - 1$.

Case 4a: $h \geq 5$, $p = 2$:

Choices $C = (0, h)$, $C = (1, h)$, and $C = (2, h)$ are ruled out because they contain at least two lattice points, namely, $(1, 1)$ and $(1, 2)$. Choices $C = (j, h)$ for $j > 2$ are ruled out because they contain at least two lattice points, namely, $(1, 1)$ and $(1, 2)$.

Case 4b: $h \geq 5$, $p = 1$:

By Pick's Formula, we have $ph = 2A = b + 2g - 2$, so $b = h$ since $p = 1$ and $g = 1$. Since $h \geq 5$, this means that there are at least 2 lattice points on the boundary of K, not counting any of the three vertices. They are certainly not on AB which has length 1, so they must be on one of the other two sides. But that would mean that AB is not the side of the triangle of shortest lattice length. Thus we have already counted such triangles somewhere above. $\square$

We note that the five lattice triangles listed are all inequivalent, because their sides have different lattice lengths. The triples of lattice lengths that can occur are: 4–2–2, 3–3–1, 2–1–1, 3–3–1, and 1–1–1.

Proposition 9.2.2 (Characterization of lattice quadrilaterals with $g = 1$). *If K is a convex lattice quadrilateral with $g = 1$, then K is lattice equivalent to one of the following six quadrilaterals:*

```
. . o       . o .       o . o
o + .       o + o       . + o
o o .       . o .       o . .

 (a)         (b)         (c)



o . o       o . . .     . o .
. + .       . + o .     . + o
o . o       o . . o     o . o

 (d)         (e)         (f)
```

Figure 9.2-2

```
All convex lattice quadrilaterals
            with g=1
```

Proof. We divide the collection of such quadrilaterals into two sets – one in which the interior lattice point lies on a diagonal of the quadrilateral, and one in which the interior lattice point does not lie on a diagonal of the quadrilateral.
Case 1: The interior lattice point lies on a diagonal of the quadrilateral.

Call the quadrilateral $ABCD$, and assume that E is a lattice point on diagonal AC. Both triangles DAC and ABC have no interior lattice points, so by the previous theorem on the Characterization of lattice triangles with $g = 0$, we know the possible affine shapes for these triangles. The two triangles must be attached to each other along an edge of lattice length 2. Each triangle must be equivalent to either TRIANG$(p, 1)$ or TRIANG$(2, 2)$. We consider two cases; one in which both triangles are equivalent to TRIANG$(p, 1)$ and the other in which at least one triangle is equivalent to TRIANG$(2, 2)$. In this latter case, we may as well assume that it is $\triangle DAC$ that is equivalent to TRIANG$(2, 2)$.
Case 1a: $\triangle ACD$ is equivalent to TRIANG$(p, 1)$. In this case, we must have $p = 2$ and we can transform $\triangle DAC$ into a right triangle with D going to $(0, 1)$, A going to $(0, 0)$, and C going to $(2, 0)$.

In this case, E goes to $(1, 0)$. Point B must be somewhere below the x-axis. Since $\triangle ABC$ has no interior lattice points, from Pick's Formula we see that its area must be $b/2 + g - 1 = 1$ since $g = 0$ and since $\triangle ABC$ is presumed to be equivalent to TRIANG$(2, 1)$ so that $b = 4$. With an area of 1 and a base (AC) of length 2, the altitude must have length 1. Thus B must lie along $y = -1$. Convexity considerations limit B to the 5 lattice points from $(0, -1)$ to $(4, -1)$. Choices $B = (0, -1)$ and $B = (4, -1)$ are ruled out because the resulting figure degenerates into a triangle. Choices $B = (1, -1)$ and $B = (2, -1)$ are valid and are shown below.

```
o . .       o . .
o + o       o + o
. o .       . . o
```

Unit shears show that these figures are equivalent to the ones shown in figures 9.2-2a and 9.2-2b. Choice $B = (3, -1)$ is also valid and is shown in the following figure, but need not be included in the characterization because the resulting quadrilateral is equivalent to the one shown in figure 9.2-2a. (To see this, apply a unit shear and the appropriate reflections.)

```
o . . .
o + o .
. . . o
```

Case 1b: $\triangle ACD$ is equivalent to TRIANG$(2, 2)$. In this case, we can transform $\triangle DAC$ into a right triangle with D going to $(0, 2)$, A going to $(0, 0)$, and C going to $(2, 0)$.

If $\triangle ABC$ is equivalent to TRIANG$(2, 1)$, then as in the previous argument, TRIANG$(2, 1)$ must have area 1 and thus B must lie on the line $y = -1$. Convexity considerations leave just 4 locations for B. The first and last are ruled out because the resulting figure degenerates into a triangle. The other two cases, $B = (1, -1)$ and $B = (2, -1)$, are valid and are both equivalent to figure 9.2-2c. (Although not obvious at first, these two figures are equivalent after performing a unit shear and the appropriate reflection.) If $\triangle ABC$ is equivalent to TRIANG$(2, 2)$, then a similar argument shows that B must line on $y = -2$, so B must vary from $(0, -2)$ to $(4, -2)$. Only $B = (2, -2)$ results in a valid quadrilateral, and this resulting parallelogram is equivalent (by a shear about the y-axis) to the rectangle shown in figure 9.2-2d.

Case 2: The interior lattice point does not lie on a diagonal of the quadrilateral.

Let the quadrilateral be $ABCD$ and draw in diagonal AC. Assume that the interior lattice point is called E and that E lies inside triangle ABC. Since $\triangle ABC$ has exactly one interior lattice point, it must be lattice equivalent to one of the five triangles shown in figure 9.2-1. Furthermore, side AC must not contain any lattice points in its interior. This immediately rules out figures 9.2-1a and 9.2-1d because each side of those triangles contain an interior lattice point.

Triangle DAC is placed against triangle ABC with one side coinciding with side AC. This side must not contain any interior lattice points, so this narrows the side down to precisely one side of each of the remaining triangles in figure 9.2-1. (All three sides of the triangle in figure 9.2-1e are equivalent, so just pick any one side.)

In each case, we find only one possible location for vertex D. Two of these (based on figures 9.2-1b and 9.2-1c) give rise to valid quadrilaterals. These are shown in figures 9.2-2e and 9.2-2f respectively. Triangle 9.2-1e yields a valid quadrilateral, but this one need not be counted since its interior lattice point happens to be contained on diagonal BD. $\square$

We note that the six quadrilaterals appearing in our characterization are all in fact inequivalent. This is because the lattice lengths of their sides differ or their interior lattice point is situated differently. The sequence of lattice lengths are: 1–1–1–1, 1–1–1–1, 2–2–1–1, 2–2–2–2, 3–2–1–1, and 2–1–1–1. The first two are inequivalent because the interior lattice point lies on both diagonals of one, but on only one diagonal of the other.

Proposition 9.2.3 (Characterization of lattice pentagons with $g = 1$). *If K is a lattice pentagon with $g = 1$, then K is lattice equivalent to one of the following three pentagons:*

```
. o .        . o .        o o .
o + o        o + o        . + o
o o .        o . o        o . o

 (a)          (b)          (c)
```

```
      Figure 9.2-3
All convex lattice pentagons
        with g=1
```

Proof. We divide the proof up into two cases, depending upon whether the interior lattice point lies on a diagonal or not.

Case 1: The interior lattice point lies on a diagonal.

Let the diagonal containing a lattice point be AD. This diagonal divides the pentagon into a quadrilateral and a triangle. Call the quadrilateral $ABCD$ so that the pentagon is named $ABCDE$. Quadrilateral $ABCD$ has one side of lattice length 2 and contains no interior lattice points. Thus it must be equivalent to

TRAP$(p, 2, 1)$ for some integer p. Map this quadrilateral into one with A going to $(0, 1)$, B going to $(0, 0)$, C going to $(p, 0)$ and D going to $(2, 1)$ and E lying above AD.

If $p = 1$, then the quadrilateral is shown below.

```
. . .
o + o
o o .
```

Triangle EAD must be equivalent to either TRIANG$(2, 1)$ or TRIANG$(2, 2)$. If $\triangle EAD$ is equivalent to TRIANG$(2, 1)$, then triangle EAD has area 1 and no interior lattice points, so E must lie along $y = 2$. Only two possibilities arise, $E = (1, 2)$ and $E = (2, 2)$. The second one is shown in figure 9.2-3a. The first one is equivalent to this one by a reflection about $x = 1$ followed by a unit shear around the x-axis.

If $\triangle EAD$ is equivalent to TRIANG$(2, 2)$, then triangle EAD has area 2 and no interior lattice points, so E must lie along $y = 3$. Only 5 choices make the resulting figure convex, $(0, 3)$ through $(4, 3)$. Two of these are ruled out because the resulting polygon is not a pentagon. Two of these are ruled out because $\triangle ADE$ is not equivalent to TRIANG$(2, 2)$. We are left with the one case, $E = (2, 3)$ which results in the pentagon shown in the figure below. This pentagon is equivalent to figure 9.2-3c as can be seen by applying a unit shear along the line $y = 2$. Furthermore, this pentagon is inequivalent to both figures 9.2-3a and 9.2-3b because it contains two sides of lattice length 2 and the others do not.

```
. . o
. . .
o + o
o o .
```

If $p = 2$, then the quadrilateral is shown below.

```
. . .
o + o
o . o
```

Triangle EAD must be equivalent to either TRIANG$(2, 1)$ or TRIANG$(2, 2)$. If $\triangle EAD$ is equivalent to TRIANG$(2, 1)$, then triangle EAD has area 1 and no interior lattice points, so E must lie along $y = 2$. The only valid spot yields $E = (1, 2)$ and the resulting pentagon is shown in figure 9.2-3b. Note that figure 9.2-3b is inequivalent fo figures 9.2-3a and 9.2-3c because figure 9.2-3b has exactly one side of lattice length 2 whereas figures 9.2-3a and 9.2-3c have a different number of sides of lattice length 2.

If $\triangle EAD$ is equivalent to TRIANG$(2, 2)$, then triangle EAD has area 2 and no interior lattice points, so E must lie along $y = 3$. Only 3 possible choices for point E exist, and each is easily ruled out.

If $p = 3$, then the quadrilateral is shown below.

```
. . . .
. . . .
o + o .
o . . o
```

There is no way to locate a triangle equivalent to the triangle TRIANG$(2,1)$ or TRIANG$(2,2)$ above AD and wind up with a convex pentagon. The same remark holds true for $p > 3$. Thus we have found all the pentagons with one interior point lying on a diagonal.

Case 2: The interior lattice point does not lie on a diagonal.

Suppose the pentagon is called $ABCDE$ and draw in diagonal AC. We claim that, F, the interior lattice point, must lie inside quadrilateral $ACDE$. For suppose that interior lattice point F lies inside triangle ABC. Then $AFCDE$ would be a convex lattice pentagon, so would contain an interior lattice point, by the Lattice Pentagon Theorem. This is a contradiction because it would show that the original pentagon contained two interior lattice points. Thus we may assume that F lies inside quadrilateral $ACDE$. Since F does not lie on any diagonal of pentagon $ABCDE$, it can not lie on any diagonal of quadrilateral $ACDE$. But there are only 2 quadrilaterals containing just one lattice point not on a diagonal of the quadrilateral. These are shown in figures 9.2-3f and 9.2-3g.

There are just 3 valid places for point B having fixed quadrilateral $ACDE$. These 3 points do not yield new pentagons, because in each case, the resulting pentagon contains its interior lattice point on a diagonal, and has already been counted. $\square$

We note that the 3 pentagons described in the characterization are all in fact inequivalent. This is because the lattice lengths of their sides differ. The sequences of lattice lengths are: 1–1–1–1–1, 2–1–1–1–1, and 2–2–1–1–1.

Proposition 9.2.4 (Characterization of lattice hexagons with $g = 1$). *If K is a lattice hexagon with $g = 1$, then K is lattice equivalent to the following centrally symmetric hexagon:*

```
. o o
o + o
o o .
```

Figure 9.2-4

Unique convex lattice hexagon with g=1

This is the Central Hexagon Theorem and was proven in section 6.

We can summarize these results with the following theorem.

Theorem 9.2.5 (Characterization of convex lattice polygons with $g =$ **1).** *If K is a convex lattice polygon with $g = 1$, then K is lattice equivalent to precisely one of the following 15 polygons:*

```
                                     o . . .
o . . . .    o . . .    . o .        . . . .
. + . . .    . + . .    . + .        . + . .
o . . . o    o . . o    o . o        o . . o


. . o        o . . .    . o .
o + .        . + o .    o + .
. o .        o . . o    o . o


. . o        . o .      o . o        o . o
o + .        o + o      . + .        . + .
o o .        . o .      o o .        o . o


. o . .      . o .      . o o        o o .
o + o .      o + o      o + o        . + o
o o . .      o . o      o o .        o . o
```

Figure 9.2-5
All convex lattice polygons
with g=1

Section 10
Convex Configurations of Lattice Points

In this section, we will investigate the properties of the interior hull of a convex lattice polygon. (Recall that the interior hull is the convex hull of the interior lattice points.) In particular, we are interested in the question: What types of sets may occur as the set of lattice points interior to a convex lattice polygon?

For example, suppose you are given two lattice points, possibly far from each other. Is there some lattice polygon that contains precisely these two points in its interior? The answer is yes, if there are no lattice points on the segment joining the given two points. For, the two points may be mapped into $(0, 0)$ and $(1, 0)$ by an integral unimodular affine transformation since the lattice length of the line segment joining these two points is 1. We easily can find a rectangle surrounding these two points and containing these two points (and no others) as its interior lattice points. The inverse integral unimodular affine transformation then gives us a parallelogram whose interior contains precisely the original two points.

Can we find such a polygon for any set of points? We will show that the answer to this is no, even if we put some obvious restrictions on the set. Before we can give this result, we need to make some definitions about configurations of lattice points and we must study the properties of such configurations.

Definition. A set of lattice points is said to be *lattice-convex* if the set contains all the lattice points in its convex hull.

We will occasionally refer to a set of lattice points that is lattice-convex as a *convex configuration* of lattice points.

Definition. A set of lattice points is said to form a *polygonal interior configuration* (or just simply a polygonal interior) if there is some convex lattice polygon whose interior lattice points comprise this set.

A polygonal interior configuration is obviously lattice-convex. Furthermore, a set of lattice points that is not lattice-convex could not be the interior of some convex set. We are interested in knowing if all convex configurations of points must be polygonal interiors.

Before investigating this problem, we need to characterize convex configurations with small cardinality.

Proposition 10.1. *If S is a convex configuration of 2 lattice points, then S is lattice equivalent to the following set:*

```
o o
```

```
     Figure 10-1
All convex configurations
   of 2 lattice points
```

Proof. Any two distinct points in the plane colline. Use the x-axis Lemma to map these points into $(0, 0)$ and $(1, 0)$ via an integral unimodular affine transformation.

Proposition 10.2. *If S is a convex configuration of 3 lattice points, then S is lattice equivalent to one of the following sets:*

```
          o .
o o o     o o
```

```
Figure 10-2
All convex configurations
of 3 lattice points
```

Proof. If the three points are collinear, then the x-axis Lemma shows that they are lattice equivalent to the set $\{(0,0),(1,0),(2,0)\}$. If the three points are not collinear, then they form a triangle. This triangle has no lattice point in its interior, because the set is lattice-convex. By the Characterization Theorem for lattice polygons with $g = 0$, we therefore conclude that this triangle is lattice equivalent to a triangle of the form TRIANG$(p, 1)$ for some integer p. But p cannot be larger than 1, for then the segment from $(0,0)$ to $(0,p)$ would contain another lattice point, contradicting the fact that the set is lattice-convex. Thus, the triangle is equivalent to TRIANG$(1,1)$. $\square$

Proposition 10.3. *If S is a convex configuration of 4 lattice points, then S is lattice equivalent to one of the following four sets:*

```
                                         . o .
                  o . .       o o        . o o
  o o o o         o o o       o o        o . .

    (a)            (b)        (c)         (d)

                    Figure 10-3
             All convex configurations
                of 4 lattice points
```

Proof. By the same reasoning as above, if the 4 points colline, then they are lattice equivalent to figure 10-3a.

If 3 of the points colline (but not 4), then the x-axis Lemma shows that we can map the points to $A = (0,0)$, $B = (1,0)$, $C = (2,0)$ with D above the x-axis. In this case, ACD forms a triangle with no interior lattice points. By the Charcterization Theorem for triangles with no interior lattice points, we know that it must be equivalent to some triangle of the form TRIANG$(p, 1)$. But side AC contains an interior lattice point and only one side of TRIANG$(p, 1)$ contains interior lattice points. Thus these two sides must map to each other and p must equal 2. The triangle then is equivalent to TRIANG$(2, 1)$ and the resulting configuration is shown in figure 10-3b.

Finally, suppose no 3 of the points colline. If the convex hull of the 4 points is a quadrilateral, then the 4 points form a quadrilateral with no interior lattice points. By the Characterization Theorem for quadrilaterals with $g = 0$, we find that the quadrilateral must be equivalent to some trapezoid of the form TRAP$(p, q, 1)$. The only such trapezoid with no lattice points on the interior of any side is TRAP$(1, 1, 1)$ which is the square shown in figure 10-3c. On the other hand, if the convex hull of the 4 points is a triangle, let D be the point inside the convex hull and let A and B be any other two of the points. Then the fourth point, C, is not inside triangle ABD, so by the previous theorem, ABD is equivalent to the triangle shown in figure 10-2. We can therefore map D to $(0,0)$, A to $(0,1)$, and B to $(1,0)$. Under this mapping, where can the fourth point C be mapped to? Point C must be mapped to some place below the x-axis, otherwise D would not be in the convex hull of $\triangle DAB$. Consider triangle DBC. It has $g = 0$ and $b = 3$, so by Pick's Formula, $A = b/2 + g - 1 = 1/2$. Since DB has length 1, this means that the altitude from vertex D must be 1, and consequently C must lie on the line $y = -1$. Now C can't be at $(-2, -1)$ or to the left of that point, for then $(-1, 0)$ would lie on or inside $\triangle ACD$. Similarly, C can't be at $(4, -1)$ or to the right of that point, for then $(2, 0)$ would lie on or inside $\triangle ABC$. Thus C must have coordinates $(x, -1)$ where $-1 \le x \le 3$. Choices $x = 0$ and $x = 2$ are ruled out because then we would have 3 of the points being collinear. Choices $x = 1$ and $x = 3$ are ruled out because then D would not be in the interior of $\triangle ABC$. Thus we find that x must be -1 and the resulting configuration is equivalent to the one shown in figure 10-3d. □

Proposition 10.4. *If S is a convex configuration of 5 lattice points, then S is lattice equivalent to one of the following seven sets:*

```
                                . o . .
                o . . .         . o o o
o o o o o       o o o o         o . . .


o .             . . o           . o .       o . .
o o             o o .           o o o       o o o
o o             o o .           . o .       o . .
```

Figure 10-4
All convex configurations
of 5 lattice points

The proof of this result is similar to the preceding proof, so we only give a sketch of the proof. The following figure shows each of the convex configurations of 4 points marked as circles. Surrounding these configurations are letters denoting the only possible locations for a fifth point. Two lattice points marked with the same letter yield configurations that are lattice equivalent.

```
                g . . . . . .
b b b b b b     d o d . . . .
a o o o o a     b o o o b . .
b b b b b b     c g e f e g c


                . . c . .
e d d e         . . o . .
d o o d         . e o o c
d o o d         . o e . .
e d d e         c . . . .
```

Theorem 10.5. *If S is a convex configuration of lattice points with cardinality less than 5, then S forms a polygonal interior configuration.*

That is, S is precisely the set of interior points for some lattice polygon.

Proof. We have completely characterized all convex configurations of 4 or fewer points. So we need only exhibit, for each possibility, a convex polygon containing that configuration in its interior. This is shown in figure 10-5, where the configuration is marked with plus signs and the vertices of the enclosing polygon are shown with circles. □

```
                            . o . .    . . . . .
o . .          o . . .      . + . .    o . . . .
. + o          . + + o      . + + o    . + + + o
o . .          o . . .      o . . .    o . . . .


                                       . . o . .
               o . . . .    o . . o    . . + . .
o . . . . .    . + . . .    . + + .    . . + + o
. + + + + o    . + + + o    . + + .    . + . . .
o . . . . .    o . . . .    o . . o    o . . . .
```

Figure 10-5
Polygons containing all convex
configurations of 4 or fewer points

Theorem 10.6. *The set* $\{(0,0),(1,0),(2,0),(3,0),(0,1)\}$ *is not a polygonal interior.*

In other words, no lattice polygon contains this set as its set of interior lattice points.

```
o . . .
o o o o
```

Figure 10-6
Lattice Point Set that is not the
interior of some convex Lattice Polygon

Proof. Let the set in question be called S. Place it so that the bottom left point lies at the origin. Suppose there were some lattice polygon, P, containing S as its set of interior lattice points. Applying the same reasoning as before, we see that all vertices of P that are below S must be on or above the line $y = -1$. Similarly, all vertices of P that are to the left of S must be on or to the right of the line $x = -1$. Finally, consider the line L_1 joining $(0,1)$ to $(3,0)$. There must be some vertex of P above this line. It must also be on or below the line L_2 joining $(1,1)$ and $(4,0)$. Let the line $x = -1$ meet lines L_1 and L_2 at points A and B respectively. Let L_2 meet the y-axis at point C and let $D = (0,1)$. Then we have shown that some vertex of P must lie in the interior of quadrilateral $ABCD$. But there are no lattice points inside this quadrilateral, so we are done. $\square$

Section 11
Realizability of Polygonal Shapes

In this section, we are concerned with the following:

Question. *What shape polygons may exist in the lattice?*

It has long been known that the square is the only regular polygon whose vertices are lattice points in the plane (Hadwiger, Debrunner, and Klee [41], page 4). We generalize this result by finding necessary and sufficient conditions for a polygon with a given shape to be realizable in the lattice, Z^2.

Definition. A polygon is said to be *realizable* in the lattice if there is a lattice polygon similar to the given polygon.

Section 11.1.
Realizability of Lattice Angles.

Proposition 11.1.1. *If A, B, and C are lattice points, then the tangent of angle ABC is rational.*

In this section, we will consider an angle whose tangent does not exist (equals $\pm\infty$) to have a rational tangent.

Proof. Let BD be the horizontal ray extending from B to the right. Then, using signed angles, we have $\angle ABC = \angle ABD + \angle DBC$. But angles ABD and DBC have rational tangents, so by applying the formula for the tangent of the sum of two angles, we see that $\angle ABC$ also has rational tangent. If either $\angle ABD$ or $\angle DBC$ is a right angle, we use the formula $\tan(x + \pi/2) = -1/\tan x$ instead. $\square$
Another proof finding an explicit formula for $\tan \angle ABC$ is useful.

Proof. Let K be the area of $\triangle ABC$ and let $\angle ABC = \theta$. Combining the law of cosines with the fact that $K = \frac{1}{2} AB \cdot BC \sin\theta$ yields

$$\tan\theta = \frac{4K}{AB^2 + BC^2 - AC^2}.$$

But the square of the length of any lattice segment is rational, and the area of any lattice triangle is rational, consequently, $\tan\theta$ is rational. $\square$

It will be useful to know what angles have rational tangents. We state some known results that we will use in the next section. For proofs, see, for example, Pólya and Szegö [81], problem 197.

Result 11.1.2. *If m and n are integers ($n \neq 0$), and if $\cos m\pi/n$ is rational, then*

$$\cos\frac{m}{n}\pi \in \{1, \frac{1}{2}, 0, -\frac{1}{2}, -1\}.$$

Result 11.1.3. *If m and n are integers ($n \neq 0$), and if $\tan^2 m\pi/n$ is rational, then*

$$\tan^2\frac{m}{n}\pi \in \{0, \frac{1}{3}, 1, 3, \infty\}.$$

For purposes of this result, we regard infinity as rational. This is equivalent to saying that $\cos^2 \frac{m}{n}\pi \in \{1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0\}$.

Result 11.1.4. *If n is a positive integer and $\tan 2\pi/n$ is rational, then $n = 1$, 2, 4, or 8.*

Section 11.2.
Realizability of Regular Lattice Polygons.

Proposition 11.2.1. *The only regular polygon that exists in the lattice is the square.*

Proof. Since the sum of the exterior angles of any polygon is 2π, each exterior angle of a regular n-gon has measure $2\pi/n$. By Result 11.1.4, n would have to be 4 or 8 otherwise this exterior angle would not have a rational tangent, so would not be realizable. The case $n = 4$ is clearly possible. If $n = 8$, let the regular octagon be $ABCDEFGH$. Then $\angle BAC = \pi/8$. This angle is not realizable because it does not have a rational tangent by Result 11.1.4. Thus the regular octagon is not realizable. $\square$

Section 11.3.
Realizability of Lattice Triangles.

Proposition 11.3.1. *If the tangents of each of the angles of $\triangle ABC$ is rational, then there is a lattice triangle similar to triangle ABC.*

Proof. If any angle of $\triangle ABC$ is a right angle, the result is obvious, so let us assume that no angle is a right angle. Suppose $\tan A = p/q$ and $\tan B = r/s$ where p, q, r, and s are non-zero integers. Locate point A at the origin, locate B at $(qr + ps, 0)$, and locate C at (qr, pr). Let D be the foot of the perpendicular from C to AB. Then $AD = qr$, $DB = ps$, and altitude $CD = pr$. Then $\tan A = pr/qr = p/q$ and $\tan B = pr/ps = r/s$, so we have constructed a lattice triangle ABC similar to the one given. $\square$

Section 11.4.
Realizability of Lattice Polygons.

Lemma 11.4.1. *Let L be any lattice line and let H be one of the open half planes determined by L. If ABC is a triangle whose angles have rational tangents, then there is a lattice triangle similar to $\triangle ABC$ with AB lying along line L and C lying in H.*

Proof. Since L is a lattice line, it must contain two lattice points. Call one of them A and the other P. We can always pick P such that if C is any point in halfplane H, then $\triangle APC$ is oriented in the same direction as the given triangle. For if P is on the wrong side of A, we can take the reflection of P around A as the new P. Note that if Q is on L and AQ is an integral multiple of AP, then Q is necessarily a lattice point.

As before, let $\tan A = p/q$ and $\tan B = r/s$. Let $AP = k$. Lay off qr copies of AP along ray AP to get to a point D and then lay off another ps copies to get to a point B. In other words, $AD = qrk$ and $DB = psk$. D and B are clearly lattice points. Erect a ray perpendicular to L at point D pointing into halfplane H. Along this ray, lay off pr copies of AP (in other words, proceed out a distance prk) to reach a point C. Let C' be a point on L such that $DC' = prk$. Then C' is a lattice point. Consequently, C is also a lattice point, because it is obtained from C' by the integral unimodular affine transformation consisting of a rotation about D through an angle of $\pi/2$.

As before, we immediately see that angles BAC and CAB have the requisite tangents, so C is in fact the desired point. $\square$

Definition. Let K be a convex polygon. A *triangularization of K by diagonals* is formed by a set of diagonals of K with the properties that

a. No two diagonals intersect except at a vertex of K.
b. K is partitioned into a union of triangles.

Lemma 11.4.2. *Let K be a convex polygon. Then in any triangularization of K by diagonals, there is some vertex of K that does not contain any of the diagonals of the triangularization.*

Proof. Let AB be the shortest diagonal of the triangularization. A and B cannot be consecutive vertices, for then AB would be a side and not a diagonal of the polygon. Diagonal AB, divides the polygonal area into two pieces. Call the smaller of these two pieces H. Thus, there is some vertex C between A and B in H. Suppose there were some diagonal, CD emenating from C. If D were not in H, CD would have to cross AB, contradicting the definition of a triangularization by diagonals. If D were in H, then CD would be smaller than AB, contradicting the minimality of AB. Thus C does not contain any of the diagonals of the triangularization. $\square$

Theorem 11.4.3. *Let K be a convex polygon. Then there is a lattice polygon similar to K if and only if for some triangularization of K by diagonals, the angles of each triangle so formed have rational tangents.*

Proof. Since all triangularizations of K by diagonals yield lattice triangles, each of the angles of these triangles must have rational tangent.

Conversely, suppose K is an n-gon and some triangularization of K by diagonals yields triangles all of whose angles have rational tangents. We will proceed by induction on n. We have already shown that the theorem is true for triangles, so suppose we have shown it to be true for all $(n-1)$-gons; we will now show it to be true for an arbitrary convex n-gon, K.

By lemma 11.4.2, there must be some vertex of K, call it C which has no diagonal emenating from it that is part of the triangularization. Let the vertices adjacent to C be called A and B.

Upon removing C from the polygon, K, we obtain a convex $(n-1)$-gon, K'. This polygon has a triangularization by diagonals wherein all triangles formed have angles with rational tangents. Thus there is a lattice $(n-1)$-gon similar to K'. Consider this $(n-1)$-gon now and let L be the line joining A' and B', the images of A and B. Let H be the halfplane determined by L that does not contain K'. By lemma 11.4.1, there is a point C in H, such that $A'BC$ is a lattice triangle similar to $\triangle ABC$ with B on L. Since $A'B$ and $A'B'$ are both lattice segments on a lattice line, their ratio is rational, say p/q. Then expand the polygon by a factor of q and the triangle by a factor of p to make their sides coincide. This therefore constructs an n-gon similar to the given one. $\square$

Section 11.5.
Realization of Oblique Pythagorean Triangles.

Definition. A *Pythagorean Triangle* is a right triangle with integer sides.

It is obvious that any Pythagorean triangle can be realized in the lattice. In general, the Pythagorean triangle with sides a, b, c, $a < b < c$ can be embedded in the lattice as TRIANG(a, b). In this embedding, the legs of the triangle are parallel to the coordinate axes.

Definition. We call a triangle *oblique* (or say that it is embedded in an oblique manner), if no side is parallel to one of the coordinate axes.

Problem. *Is there an oblique lattice triangle similar to a 3-4-5 right triangle?*

Solution. The answer is yes. A computer search finds that the smallest such triangle has vertices at $(0,0)$, $(3,6)$, and $(11,2)$.

We note that the sides of this triangle have length $3\sqrt{5}$, $4\sqrt{5}$, and $5\sqrt{5}$. A more interesting question is: Can we find such a triangle with integral sides?

We can answer this question in the affirmative by using lemma 11.4.1.

Let L be the lattice line passing through the origin, O, and the lattice point $A = (m, n)$ in the first quadrant. By the lemma, we can find a lattice triangle similar to a 3-4-5 triangle and having its smallest side lying on L and its third point in the upper half plane. Applying the algorithm specified by the lemma, we extend OA by twice its length to reach the point $B = (3m, 3n)$. Rotate BO counterclockise around B through an angle of $\pi/2$, bringing A into the point $A' = (3m - 2n, , 3n + 2m)$. Extend BA' its own length past A' to reach the point $C = (3m - 4n, 3n + 4m)$. Then since $BA = BA'$ and $BA = 2AO$ and $BC = 2BA'$, we therefore find that $\triangle OBC$ is similar to a 3-4-5 right triangle.

We have

$$\begin{aligned} O &= (0,0) \\ B &= (3m, 3n) \\ C &= (3m - 4n, 3n + 4m) \end{aligned}$$

We note that letting $m = 2$ and $n = 1$ yields the triangle previously found by the computer search. This algorithm also provides us with a general method of finding such triangles.

To make the sides of the triangle integral, we first make OB integral. To do this, we apply the general formula for the sides of a Pythagorean triangle and let $m = p^2 - q^2$ and $n = 2pq$. This makes OA integral of length $p^q + q^2$. Point C now has coordinates $(3p^2 - 3q^2 - 8pq, 6pq + 4p^2 - 4q^2)$. OB and BC are necessarily integral because $OB = 3OA$ and $BC = 4OA$. OC is necessarily integral because $\triangle OBC$ is similar to a 3-4-5 triangle and OB is integral. Clearly, $OC = 5OA$.

We thus have a two-parameter family of triangles similar to a 3-4-5 right triangle:

$$\begin{aligned} O &= (0,0) \\ B &= (3p^2 - 3q^2, 6pq) \\ C &= (3p^2 - 3q^2 - 8pq, 6pq + 4p^2 - 4q^2) \end{aligned}$$

In some of these, a side may be parallel to one of the axes. It is simple to avoid such a case. For example, choose $p = 4$ and $q = 1$ to get the integral triangle with vertices at $(0,0)$, $(45,24)$, and $(13,84)$. This triangle has sides of lenghts 51, 68, 85. It is 17 times as large as a 3-4-5 triangle.

Another 2-parameter colution can be obtained by letting $m = 2pq$ and $n = p^2 - q^2$ instead. This yields

$$
\begin{aligned}
O &= (0,0)\\
B &= (6pq, 3p^2 - 3q^2)\\
C &= (6pq - 4p^2 + 4q^2, 3p^2 - 3q^2 + 8pq)
\end{aligned}
$$

A computer search reveals that the smallest integral triangle similar to a 3-4-5 triangle with no side parallel to an axis, is the triangle with vertices at $(0,0)$, $(36,15)$, and $(16,63)$. It has sides of lengths 39, 52, 65 and is 13 times as large as a 3-4-5 triangle. This triangle can be obtained from our second 2-parameter solution by letting $p = 3$ and $q = 2$.

Proposition 11.5.1. *Given a Pythagorean Triangle, one can find an oblique Pythagorean lattice triangle similar to the given triangle.*

Proof. Suppose the given Pythagorean triangle has sides r, s, and t, with t being the length of the hypotenuse. Using the same method as before, let $A = (m,n)$. Lay off r copies of OA along ray OA to bring us to the point $B = (rm, rn)$. Erect a perpendicular to OB at B and lay off s copies of OA to bring us to the point $C = (rm - sn, rn + sn)$.

Now let $m = p^2 - q^2$ and $n = 2pq$ to guarantee that OA has integral length. Then we have constructed a Pythagorean triangle OBC similar to the given triangle. OB and BC are clearly not parallel to any axis. OC might be parallel to the y-axis. To prevent this, pick $p = 4s$ and $q = 1$. Then the sides of the resulting triangle are

$$
\begin{aligned}
O &= (0,0)\\
B &= (16rs^2 - r, 8sr)\\
C &= (16rs^2 - r - 8s^2, 8rs + 8s^2).
\end{aligned}
$$

The line OC cannot be parallel to the y-axis, for that would require $16rs^2 = r + 8s^2$ or $s^2 = r/8(2r-1) \leq (2r-1)/8(2r-1) = 1/8$ which cannot happen since s^2 is a positive integer. $\square$

Recall that a Pythagorean Triangle is called *primitive* if its three sides are relatively prime.

The above procedure always produces a non-primitive Pythagorean triangle, since all sides of the triangle formed are divisible by the length of OA and it is clear that $OA \neq 1$. It is therefore natural to ask if there is a primitive Pythagorean triangle embedded obliquely in the lattice. We answer this question in the negative.

Theorem 11.5.2. *No primitive Pythagorean triangle can be embedded obliquely in the lattice.*

Proof. Suppose Pythagorean triangle ABC (with right angle at C) is embedded obliquely in the lattice. Translate the triangle so that C coincides with the origin. Then perform a rotation through a multiple of $\pi/2$ until ray CB lies in the first quadrant. Point B cannot wind up on an axis since the triangle is still embedded obliquely, this property not being disturbed by the translations or rotations just performed. We may assume that point A has gone into the second quadrant, for if it went into the third quadrant, we may perform a reflection about the line $y = x$ to bring it into the second quadrant, leaving B in the first quadrant. Furthermore, we may assume that B lies further from the x-axis than A, for if A had a larger ordinate, we could perform a reflection about the y-axis and then relabel points A and B.

Let D be the foot of the perpendicular from B to the x-axis, and let E be the foot of the perpendicular from A to BD. Since B was further than A from the x-axis, point E lies between B and D. Also note that since A and B are lattice points, the coordinates of points A, B, D, and E are integers. Quadrilateral $ACEB$ is cyclic because $\angle ACB = \angle AEB = \pi/2$. Thus $\angle ABC = \angle AEC$. But $AE \parallel CD$ implies that $\angle AEC = \angle ECD$. Thus $\angle ABC = \angle ECD$. But triangles ECD and ABC are right triangles. Hence they are similar. Let the ratio of similarity be p/q with $\gcd(p, q) = 1$. This ratio is rational because it is equal to the ratio of DE to AC, both of which are integral. But $AB > BC > CE$, so $\triangle ABC$ is strictly larger than $\triangle CDE$ and so $q > 1$. Now $CE = (p/q) \cdot AB$, so CE is rational. But $CE^2 = CD^2 + DE^2$, so CE^2 is an integer. If a rational number squared is integral, the rational number must itself be an integer. Hence BC is an integer. Let the lengths of the sides of $\triangle ABC$ be a, b, and c. Then the lengths of the sides of $\triangle ECD$ are pa/q, pb/q, and pc/q. But these lengths are integers and p and q are relatively prime. So $q|a$, $q|b$, and $q|c$. Thus $q| \gcd(a, b, c)$ and consequently, $\triangle ABC$ is not primitive. $\square$

Corollary 11.5.3. *The set of diophantine equations*

$$\begin{aligned} a^2 + b^2 &= r^2 \\ (b+d)^2 + c^2 &= s^2 \\ (a+c)^2 + d^2 &= t^2 \\ r^2 + s^2 &= t^2 \end{aligned}$$

has no solution with r, s, and t being relatively prime.

Proof. In the preceding configuration, let point B have coordinates (c, d), let C have coordinates $(-a, b+d)$ and let $AC = r$, $AB = s$, and $BC = t$. Then the above equations represent the Pythagorean Theorem applied to the various right triangles involved. $\square$

Section 11.6.
Realizability of Equiangular Lattice Polygons.

Recall that a polygon is said to be *equiangular* if all of its interior angles are equal.

It is known that equiangular n-gons exist in the lattice if and only if $n = 4$ or $n = 8$. See Honsberger [60] or Pólya and Szegö [81], problem 238.1.

Proposition 11.6.1. *Equiangular lattice n-gons exist if and only if $n = 4$ or $n = 8$.*

Proof. Since the sum of the exterior angles of a polygon is 2π, for an equiangular n-gon, each exterior angle must have measure $2\pi/n$. By Result 11.1.4, this angle can have a rational tangent if and only if $n = 4$ or $n = 8$. Both cases can occur as can be seen by figure 11.6-1. $\square$

```
                . o o .
                o . . o
  o o           o . . o
  o o           . o o .
```

Figure 11.6-1
Equiangular lattice n-gons

for n=4 and n=8
with smallest g

Corollary 11.6.2. *The interior angle of an equiangular lattice polygon must be $\pi/2$ or $3\pi/4$.*

There are equiangular lattice polygons other than the ones shown above. For example:

```
. . . . o .
. . . . . .        . o . . o .
o . . . . o        o . . . . o
. . . . . .        o . . . . o
. o . . . .        . o . . o .
```

Figure 11.6-2
Other equiangular lattice polygons

We note that all lattice rectangles are equiangular and all equiangular polygons are necessarily convex.

Section 11.7. Realizability of Equilateral Convex Lattice Polygons.

Recall that a polygon is said to be *equilateral* if all of its sides have the same length.

Proposition 11.7.1. *There is no equilateral lattice n-gon if n is odd.*

This proposition was first proven by Dean Hoffman.

Proof. (Honsberger [60]) We proceed by contradiction. Suppose there are equilateral lattice n-gons for odd n. Let d be the length of the side of the smallest of these polygons; and suppose this polygon has n sides with n odd.

Two integers are said to have the same parity if they are both even or both odd.

Since d^2 is the sum of two squares, d^2 must be congruent to 0, 1, or 2 modulo 4, since no square can be congruent to 3 (mod 4).

Case 1. $d^2 \equiv 0 \pmod 4$.

Let (x_1, y_1) and (x_2, y_2) be any two consecutive vertices of the polygon. Let $p = x_2 - x_1$ and $q = y_2 - y_1$, so that $d^2 = p^2 + q^2$. Since d is even, p and q must have the same parity. If p and q were both odd, then d^2 would not be divisible by 4. Thus p and q must both be even. Then since the coordinates of all the vertices are even, we could scale the polygon down by a factor of 2 and get another equilateral lattice polygon. This contradicts the minimality of d.

Case 2. $d^2 \equiv 1 \pmod 4$.

Color each vertex, (x, y), red if x and y have the same parity; otherwise color the vertex blue. Since there are an odd number of vertices in the polygon, some two adjacent vertices (x_1, y_1) and (x_2, y_2) would have to have the same color. But then $x_1 + y_1 \equiv x_2 + y_2 \pmod 2$, so $x_2 - x_1 \equiv y_2 - y_1 \pmod 2$ and $d^2 \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2 \equiv 0 \pmod 2$, contradicting the fact that $d^2 \equiv 1 \pmod 4$.

Case 3. $d^2 \equiv 2 \pmod 4$.

Color each vertex, (x, y), red if x is even and color it blue if x is odd. Since there are an odd number of vertices in the polygon, some two adjacent vertices (x_1, y_1) and (x_2, y_2) would have to have the same color. This implies that $x_2 - x_1$ is even. But $d^2 \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2 \equiv 2 \pmod 4$; and the only way the sum of two squares can be congruent to 2 (mod 4) is if each square is congruent to 1 (mod 4). This contradicts the fact that $x_2 - x_1$ is even.

We have thus reached a contradiction in each case. ☐

Hoffman has also shown that equilateral n-gons exist for all even n larger than 2. His proof, however, shows the existence of such an n-gon, but the n-gon he produces need not be convex. We will prove the stronger result that convex equilateral n-gons exist for all even n, $n > 2$.

The proof of the existence of convex equilateral n-gons for even n requires some results from elementary number theory.

Lemma 11.7.2. *The number of ordered integral solutions (x, y) of the equation $x^2 + y^2 = m$, where m is a positive integer is $4(A - B)$ where A is the number of divisors of m of the form $4k + 1$ and B is the number of divisors of m of the form $4k + 3$.*

For a proof, see Keng [64], p.120. Note that this includes negative integral values for x and y.

Lemma 11.7.3. *If n is any positive integer, then there exists a positive integer m, such that the equation $x^2 + y^2 = m$ has at least n positive integral solutions.*

This follows immediately from lemma 11.7.2 since we can pick $m = 5^{2n}$ and notice that the positive and negative solutions come in pairs.

Proposition 11.7.4. *Convex equilateral lattice n-gons exist for all even n greater than 2.*

Proof. Let $n = 2k$ be an even integer larger than 2. By lemma 11.7.3, there is a positive integer, N, that can be written as the ordered sum of two squares, $a^2 + b^2$, $a, b \in Z^+$, in at least k different ways. For example, 50 is the smallest positive integer that can be written as the sum of two positive squares in 3 ways, namely, $50 = 1^2 + 7^2 = 5^2 + 5^2 = 7^2 + 1^2$.

Order these k representations such that the ratio a/b increases. Then, beginning at the origin, we may lay off k successive segments heading into the first quadrant, with slopes of a/b. These segments form a convex lattice polygon $OA_1A_2A_3 \ldots A_k$.

Let B_k be the reflection of the point A_k about the x-axis. In that case, $A_kA_{k-1} \ldots A_2A_1OB_1B_2 \ldots B_{k-1}B_k$ is a convex lattice polygon with $2k$ sides, not counting A_kB_k. Reflect this polygon about A_kB_k and we wind up with a convex equilateral lattice polygon with $4k$ sides.

Also note that $A_kA_{k-1} \ldots A_2A_1OB_1B_2 \ldots B_{k-1}$ is a convex lattice polygon with $2k - 1$ sides, not counting A_kB_{k-1}. Reflect this polygon about A_kB_{k-1} and we wind up with a convex equilateral lattice polygon with $4k - 2$ sides.

We have thus shown how to construct a convex equilateral lattice polygon with n sides for any even n, $n > 2$. ☐

Small convex equilateral n-gons for $n = 2$, 4, 6, 8, 10, and 12 are shown in figures 11.7-1, 11.7-2, and 11.7-3. Note that these polygons were not constructed by the procedure outlined in the preceding proof. (That procedure creates much larger polygons, in general.)

```
                                . . . o . . .
                                . o . . . o .
             . . . o . .        . . . . . . .
             . o . . . o        o . . . . . o
             . . . . . .        . . . . . . .
o o          o . . . o .        . o . . . o .
o o          . . o . . .        . . . o . . .
```

figure 11.7-1
Convex equilateral lattice n-gons
for n=4, 6, and 8

```
. . . . . . . o . . . . o . . . .
. . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . .
. . . o . . . . . . . . . . . . o
. . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . .
o . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . o
. . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . .
o . . . . . . . . . . . . o . . .
. . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . .
. . . . o . . . . o . . . . . . .
```

figure 11.7-2
Convex equilateral lattice decagon

```
. . . . . . . o . . . . o . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . o . . . . . . . . . . . . o . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
o . . . . . . . . . . . . . . . . . . o
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
o . . . . . . . . . . . . . . . . . . o
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . o . . . . . . . . . . . . o . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . .
. . . . . . . o . . . . o . . . . . . .
```

Figure 11.7-3
Convex equilateral lattice dodecagon

Corollary 11.7.5. *Central symmetric lattice n-gons exist if and only if n is even.*

Proof. Central symmetry clearly implies that n must be even. If n is even, the preceding construction not only produced a convex equilateral n-gon, but also produced one that was centrally symmetric. □

Figure 11.7-4 shows convex central symmetric n-gons for $n = 4$, 6, 8, and 10, with the smallest number of interior lattice points. The fact that g is minimal follows from the theorems that say $v = 6 \Rightarrow g \geq 1$, $v = 8 \Rightarrow g \geq 4$, and $v = 10 \Rightarrow g \geq 10$. Again, these polygons were not constructed by the procedure given in the preceding proof, but rather were picked to have the smallest g.

```
                                    . . . . o o
                                    . . o . . o
                      . o o .       . o . . . .
          . o o       o . . o       . . . . o .
o o       o . o       o . . o       o . . o . .
o o       o o .       . o o .       o o . . . .
```

figure 11.7-4
Convex central symmetric lattice n-gons
for n=4, 6, 8, and 10
with smallest g

Section 11.8.
Realizability of Cyclic Lattice Polygons.

Recall that a polygon is said to be *cyclic* if all of its vertices lie on a circle and it is said to be *circumscribable* if all of its sides are tangent to a common circle.

Proposition 11.8.1. *A cyclic lattice n-gon exists for all $n \geq 3$.*

Proof. By lemma 11.7.3, for any n we can find an R and n ordered pairs of integers (a, b) such that $a^2 + b^2 = R^2$. These n points lie on a circle of radius R about the origin and thus form the desired cyclic n-gon. This cyclic n-gon is necessarily convex. □

```
                  o . . .     o . . o
o .     o o       o . . o     o . . o
o o     o o       . o o .     . o o .


. o . .     . o o .
o . . o     o . . o
o . . o     o . . o
. o o .     . o o .
```

figure 11.8-1
Cyclic lattice n-gons
for n=3, 4, 5, 6, 7, and 8
and smallest g

Proposition 11.8.2. *A circumscribable lattice n-gon exists for all $n \geq 3$.*

Proof. By the previous proposition, a cyclic n-gon, $A_1A_2A_3 \dots A_n$ with center at the origin, O, exists. Furthermore, by the previous construction, we can make sure that at least one point lies in each quadrant of the circle. (If $n = 3$, it is obvious that a circumscribable lattice polygon exists.) At each vertex of this polygon, draw a tangent to the polygon's circumcircle. These tangents meet their neighboring tangents in n points forming a convex polygon $B_1B_2B_3 \dots B_n$. The restriction that there is at least one point in each quadrant guarantees that these lines meet on the correct side of the circle and that no two successive ones are parallel. Since OA_i has rational slope, each of these tangents has rational slope. Thus each B_i has rational coordinates. Perform a similarity about the origin, expanding by the least common multiple of the denominators of these rational coordinates and the resulting polygon will be a circumscribable lattice polygon. This circumscribable n-gon is necessarily convex. □

Section 11.9.
Realizability of Integral Polygons in the Lattice.

We recall that an *integral polygon* is a polygon with integer sides and a *Heronian polygon* is a polygon with integer sides and integral area.

A well-known construction shows how to construct an integral n-gon in the lattice for all n. In fact, a stronger result is obtained.

Proposition 11.9.1. *For all positive integers $n \geq 3$, there exists a convex lattice n-gon all of whose sides and diagonals are integral.*

Proof. (Gleason, Greenwood, and Kelly [37], p. 470]) Pick θ so that both $\cos\theta$ and $\sin\theta$ are rational. The points $\{e^{2mi\theta}\}, m = 1, 2, \dots, n$, are all at rational distances from one another. By an appropriate change of scale, we can make all of these distances integral. □

Corollary 11.9.2. *For all positive integers $n \geq 3$, there exists a convex Heronian lattice n-gon.*

Proof. This is an immediate consequence since the area of any lattice polygon is a multiple of 1/2. If the previously constructed polygon does not have integral area, merely scale it up by a factor of 2. □

Section 11.10.
Known Results in E^n.

For the interested reader, we summarize similar known results in E^n.

It has long been known (Schoenberg [94], Chrestenson [22], Pólya and Szegö [81], problem 244.1) that the only regular polygons that can be embedded in the cubic lattice of E^n are the square (for $n \geq 2$), the triangle and the hexagon (both for $n \geq 3$).

Various authors have investigated which regular polytopes can be embedded in the cubic lattice of E^n, but the study was incomplete until Greg Patruno ([80]) settled the question for cubic lattices and all other regular polytopal lattices. We summarize the final results for the standard cubic lattice below.

To describe the polytopes, we use the standard Schläfli notation. The symbol $\{m\}$ denotes a regular m-gon. The regular n-dimensional polytope represented by $\{a_1, a_2, \ldots, a_{n-1}\}$ is a convex configuration of congruent $\{a_1, a_2, \ldots, a_{n-2}\}$'s called cells which fit together in such a way that each $(n-2)$-dimensional face belongs to two cells, and each $(n-3)$-dimensional edge to a_{n-1} cells. The notation 3_k represents the sequence $3, 3, \ldots, 3$ where the 3 is repeated k times.

The complete set of regular polytopes is given in Coxeter [23] pp. 292-295. They consist of

a. in E^2: $\{m\}$, for $m \geq 3$
b. in E^3: $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$
c. in E^4: $\{3,3,3\}$, $\{3,3,4\}$, $\{3,4,3\}$, $\{4,3,3\}$, $\{3,3,5\}$, $\{5,3,3\}$
d. in E^n: $\{3_{n-1}\}$, $\{3_{n-2}, 4\}$, $\{4, 3_{n-2}\}$ for $n \geq 5$.

First we note that a d-dimensional polytope cannot be embedded in E^n with $n < d$. Also, if it can be embedded in E^n, then it can be embedded in any space of dimension larger than n.

Result 11.10.1. *(regular polygons).*

a. The square, $\{4\}$, can be embedded in E^n for all $n \geq 2$. An embedding in E^2 is given by $(0, \pm 1)$, $(\pm 1, 0)$.
b. The equilateral triangle, $\{3\}$, can be embedded in E^n if and only if $n \geq 3$. An embedding in E^3 consists of the 3 permutations of $(1, 0, 0)$.
c. The regular hexagon, $\{6\}$, can be embedded in E^n if and only if $n \geq 3$. An embedding in E^3 consists of the 6 permutations of $(-1, 0, 1)$.
d. The regular m-gons, $\{m\}$, for $m \neq 3, 4, 6$ cannot be embedded in any E^n.

References.

a. Obvious.
b. Patruno [80]
c. Patruno [80]
d. Hadwiger, Debrunner, and Klee [41] p. 5

Result 11.10.2. *(regular polytopes). For $m \geq 3$,*

a. The regular m-simplex, $\{3_{m-1}\}$, can be embedded in E^n for all $n > m$. An embedding in E^{n+1} consists of the $n+1$ permutations of $(1, 0_n)$.
b. For $n = m$, the regular m-simplex, $\{3_{m-1}\}$, can be embedded in E^n if and only if m is an odd square, the sum of two odd squares, or a multiple of 4.
c. The m-cube, $\{3_{m-2}, 4\}$, can be embedded in E^n for all $n \geq m$. An embedding in E^n consists of the n permutations of $(1, 0_{n-1})$ and their negatives.
d. The cross polytope, $\{4, 3_{m-2}\}$, can be embedded in E^n for all $n \geq m$. An embedding in E^n consists of $(\pm 1_n)$. This includes the regular octahedron (when $n = 3$).
e. The regular icosahedron, $\{3,5\}$, regular dodecahedron, $\{5,3\}$, and their 4-dimensional relatives, $\{3,3,5\}$, and $\{5,3,3\}$ cannot be embedded in E^n for any n.
f. $\{3,4,3\}$ can be embedded in E^n for all $n \geq 4$. An embedding in E^4 consists of the 24 permutations of $(\pm 1, \pm 1, 0, 0)$.

References.

a. Patruno [80]
b. Schoenberg [94]
c. Patruno [80]

d. Patruno [80]
e. Patruno [80]
f. Schläfli [92]

Consult Patruno [80] for similar results for regular lattices other than the cubic lattice.

Section 12
Results implying interior lattice points

There are many known inequalities for convex bodies with no interior lattice points. In this section, we review these results and then give analogous results for convex lattice polygons.

Let K be a convex body in the plane with no lattice points in its interior. Suppose K has area A, perimeter P, diameter D, (minimal) width w, inradius r, and circumradius R.

Proposition 12.1. *If K is a convex body in the plane with no interior lattice points, then $A \leq P/2$.*

Reference. Bender [6].

Proposition 12.2. *If K is a convex body in the plane with no interior lattice points, then $(w-1)(D-1) \leq 1$ with equality when and only when K is a triangle with $w = D/(D-1)$. Equivalently, $(w-1)D \leq w$.*

Reference. Scott [106].

Proposition 12.3. *If K is a convex body in the plane with no interior lattice points, then $(w-1)A \leq w^2/2$ with equality when and only when K is a triangle and $D = w/(w-1)$.*

Reference. Scott [109].

Proposition 12.4. *If K is a convex body in the plane with no interior lattice points, then $(D-1)A \leq D^2/2$ providing $D \leq 2$ with equality when and only when K is a square of diameter $D = 2$.*

Reference. Scott [109].

Proposition 12.5. *If K is a convex body with no interior lattice points, then*

a. $w \leq (2+\sqrt{3})/2$
b. $w \leq 3r$
c. $(w-1)D \leq (2+\sqrt{3})/2$
d. $(w-1)D \leq w$
e. $(w-1)A \leq (7+4\sqrt{3})/8$
f. $(w-1)A \leq 3wr/2$
g. $(w-1)P \leq (6+3\sqrt{3})/2$
h. $(w-1)P \leq 9r$
i. $(w-1)R \leq (3+2\sqrt{3})/6$
j. $(w-1)R \leq r\sqrt{3}$
k. $(w-1)P \leq 3w$
l. $(w-1)R \leq w/\sqrt{3}$
m. $(w-1)A \leq w^2/2$
n. $(w-1)(D-1) \leq 1$

In each case, equality occurs when and only when K is an equilateral triangle of side length $(2+\sqrt{3})/\sqrt{3}$.

Proof. See Scott [107]. □

We now give analogs for lattice polygons.

Theorem 12.6. *If K is a convex lattice polygon with no interior lattice points, then*

a. $w \leq \sqrt{2}$
b. $w \leq (\sqrt{2}+1)r$
c. $(w-1)D \leq 4-2\sqrt{2}$
d. $(w-1)D \leq 2(\sqrt{2}-1)w$
e. $(w-1)A \leq 2\sqrt{2}-2$
f. $(w-1)A \leq (\sqrt{2}-1)w^2$
g. $(w-1)P \leq 2\sqrt{2}$
h. $(w-1)P \leq 2w$
i. $(w-1)R \leq 2-\sqrt{2}$
j. $(w-1)R \leq (\sqrt{2}-1)w$
k. $(w-1)P \leq 3(\sqrt{2}+1)r$
l. $(w-1)(D-1) \leq 5-3\sqrt{2}$

In each case, equality occurs when and only when K is lattice congruent to TRIANG$(2,2)$.

This theorem follows as a direct consequence of the following theorem.

Theorem 12.7. *If K is a convex lattice polygon with no interior points, then $w \leq \sqrt{2}$ and equality holds when and only when K is lattice congruent to TRIANG$(2,2)$. For all other lattice polygons, $w \leq 1$.*

This follows from our computer search of all lattice polygons with $g = 0$ and $D \leq 10$.

We also have analogs to Jung's Theorem and Blashke's Theorem:

Proposition 12.8. *If K is a convex lattice polygon with no interior lattice points, then $R \leq D/2$.*

Compare with Jung's Theorem which says that $R \leq D/\sqrt{3}$.

Proposition 12.9. *If K is a convex lattice polygon with no interior lattice points, then $w \leq (\sqrt{2}+1)r$.*

Compare with Blashke's Theorem which says that $w \leq 3r$. Both of the previous results fall out of our computer search.

Section 13
Notable Lattice Points

There are many notable points associated with a simplex in E^n, such as the incenter, circumcenter, centroid, etc. In this section, we are interested in the problem of finding lattice simplices whose notable points are also lattice points.

We start by recalling some definitions. A *rational point* is a point all of whose coordinates are rational. The centroid of an n-simplex is the point of intersection of the lines from each vertex to the centroid of the opposite facet. The *circumcenter* of a simplex is the center of the circumscribed sphere and the *incenter* of a simplex is the center of its inscribed sphere. If the altitudes of the simplex intersect, then this intersection point is known as the *orthocenter*.

It is easy to show that the centroid and circumcenter of a lattice simplex must be rational points. The orthocenter does not always exist, but when it does, it must be a rational point. If a notable point of a lattice simplex is rational, a scaling shows that we can find a lattice simplex where this notable point is a lattice point.

In general, the incenter of a lattice simplex need not be rational. We now show how to find lattice simplices with rational incenters. Many such simplices exist. We give a constructive procedure for exhibiting one.

Proposition 13.1. *In E^n, there exists a lattice simplex whose incenter is a lattice point.*

Proof. Consider the hyperplane $a_1x_1 + a_2x_2 + \cdots + a_nx_n - p = 0$. The distance from the origin to this hyperplane is

$$d = \frac{p}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}.$$

If the hyperplane meets axis x_i at a distance c_i from the origin, then we have $a_i = p/c_i$. Thus

$$d = \frac{1}{\sqrt{\frac{1}{c_1^2} + \cdots + \frac{1}{c_n^2}}}.$$

The n intercepts plus the origin yield $n+1$ points forming an n-simplex. Let T be the facet of this simplex opposite the origin. By picking each c_i to be an integer, we force all facets except T to have rational $(n-1)$-dimensional volume. To make the incenter rational, we need only make the $(n-1)$-dimensional volume of T rational, since the inradius is connected to F, the sum of the volumes of the facets, by the formula $nV = Fr$ where V is the volume of the simplex and r is the length of the inradius. The volume is clearly rational since the vertices are rational, so T will have rational volume if the altitude to T has rational length. But this length is d.

After making the incenter rational, an appropriate scaling will make it lie at a lattice point.

Thus, the problem is reduced to the number theory problem of determining n integers whose square is an integer. We show how to do this for any integer n by induction.

For $n = 2$, we obviously have $3^2 + 4^2 = 5^2$. Assume now that we have found integers a_1 through a_k whose sum of squares is a perfect square, c^2, and assume further that c is odd. We really seek integers p and q such that $c^2 + p^2 = q^2$ and q is odd. Take $p = (c^2 - 1)/2$ and $q = (c^2 + 1)/2$. $\square$

For example: using the above procedure, we can find 5 squares that sum to a square; namely, $3^2 + 4^2 + 12^2 + 84^2 + 2212^2 = 2213^2$.

Section 14
The Volume of Polytopes in E^n

In this section, we look at some results concerning the volume of polytopes in E^n.

Section 14.1.
The volume of a lattice polytope.

Let K be a convex lattice polytope in E^n. Let $G(K)$ denote the number of lattice points in the set K and let $g(K)$ denote the number of lattice points in the interior of the set K. We want to express the volume, $V(K)$ of K in terms of G and g.

In 2-space, the problem is completely solved by Pick's Formula. We give below three equivalent formulations.

Pick's Formula. *If $n = 2$, then*
a. $V(K) = G(\partial K)/2 + g(K) - 1$.
b. $V(K) = G(K) - G(\partial K)/2 - 1$.
c. $V(K) = G(K)/2 + g(K)/2 - 1$.

J. E. Reeve found an analog in 3-space. We need some notation first. If m is a positive integer, let Λ_m be the lattice of all points (a, b, c) such that ma, mb, and mc are integers. If K is a convex body, $g_m(K)$ denotes the number of points from the lattice Λ_m that are in K.

Reeve's Formula. *Let m be a positive integer and let K be a convex lattice polytope in E^3. Let $f_m(X) = g_m(X) - mG(X)$. Then, $V(K)$, the volume of K is given by*

$$2(m-1)m(m+1)V(K) = 2f_m(K) - f_m(\partial K).$$

Refrence. Reeve [83].

Theorem (Reeve). *Let m be a positive integer and let K be a convex lattice polytope in E^3. Then*

$$g_m(\partial K) - m^2 G(\partial K) = 2(1 - m^2).$$

Reference. Reeve [83].

If we allow the lattice polytope to cross itself, then we must be more careful about what we mean by the volume of the polytope, and the volume will depend on the Euler characteristic of the polytope. Interested readers should consult Reeve [83]. The result follows.

We let $\chi(K)$ denote the Euler-Poincaré characteristic of K.

Theorem (Reeve). *Let m be a positive integer. If $M_m(K) = g_m(K) - mG(K) - (m-1)\chi(K)$, then the volume, $V(K)$, of K is given by*

$$2(m-1)m(m+1)V(K) = 2M_m(K) - M_m(\partial K).$$

Reference. Reeve [83].

Theorem. *If $m \in Z$, then*

$$2g_m(K) - 2m^2G(K) + 2(1-m^2)\chi(K) + mg_m(\partial K) - mg(\partial K) = 0.$$

Reference. Reeve [83].

MacDonald generalized to E^n.

MacDonalds Formula. *If K is a lattice polytope in E^n, then the volume, V, can be calculated from the formula*

$$(n-1)n!V = M(n-1) - \binom{n-1}{1}M(n-2) + \binom{n-1}{2}M(n-3) - \ldots + (-1)^{n-1}M(0)$$

where $M(n)$ denotes $2g_n(K) - g_n(\partial K)$ if $n > 0$ and $M(0)$ denotes $2\chi(K) - \chi(\partial K)$. If K is a simple convex lattice polytope, then $M(0)$ is 2 or 0 according as n is even or odd.

Reference. MacDonald [69].

Section 14.2.
The volume of an n-simplex with many equal edges.

It is well known that the volume of a regular n-simplex with edge length s is

$$\frac{s^n}{n!}\sqrt{\frac{n+1}{2^n}}.$$

(For a proof, see Sommerville [116].) But suppose one edge has length b (and all the other edges have length a). Is there a simple formula for the volume of the simplex in that case? What if all the edges incident at a given vertex have length b and all the other edges have length a?

Theorem 14. 2.1. Let K be an n-simplex in E^n. Suppose the vertices of K are colored with r colors, c_1, $c_2, \ldots,$ c_r $(1 \le r \le n+1)$. Let the number of vertices colored c_i be m_i $(1 \le m_i \le n+1)$. It is given that if an edge has both its vertices the same color, c_i, the length of that edge is a_i. If the two vertices of an edge have different color, the edge has length s. Then the volume of K is

$$\frac{1}{n!2^{n/2}}\prod_{i=1}^{r} a_i^{m_i-1}\sqrt{(-1)^{r+1}\left(\prod_{i=1}^{r}\left((m_i-1)a_i^2-m_is^2\right)\right)\sum_{i=1}^{r}\frac{m_i}{(m_i-1)a_i^2-m_is^2}}.$$

Proof. The volume, V, of an n-simplex in terms of the edge lengths, $\{a_{ij}\}$, is determined by the formula

$$(-1)^{n+1}2^n(n!)^2V^2 = D \tag{1}$$

where D is given by the determinant

$$\begin{vmatrix} 0 & a_{12}^2 & a_{13}^2 & \cdots & a_{1n}^2 & a_{1,n+1}^2 & 1 \\ a_{21}^2 & 0 & a_{23}^2 & \cdots & a_{2n}^2 & a_{2,n+1}^2 & 1 \\ a_{31}^2 & a_{32}^2 & 0 & \cdots & a_{3n}^2 & a_{3,n+1}^2 & 1 \\ \vdots & & & & \ddots & & \\ a_{n+1,1}^2 & a_{n+1,2}^2 & a_{n+1,3}^2 & \cdots & a_{n+1,n}^2 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}.$$

(See Sommerville [116] for a proof.)

Now, let us assign the edge lengths as specified in the theorem, except that to make the computations simpler, let us assume the edge lengths are $\sqrt{a_i}$ and $\sqrt{s}$ (instead of a_i and s). A simple transformation then will change the result we get into the form required by the statement of the theorem.

We find that the resulting determinant consists of r square blocks along the main diagonal and the last row and column being the same as shown above. The ith block has the form

$$\begin{pmatrix} 0 & a_i & a_i & a_i & \cdots & a_i & a_i \\ a_i & 0 & a_i & a_i & \cdots & a_i & a_i \\ a_i & a_i & 0 & a_i & \cdots & a_i & a_i \\ a_i & a_i & a_i & 0 & \cdots & a_i & a_i \\ \vdots & & & & \ddots & & \\ a_i & a_i & a_i & a_i & \cdots & 0 & a_i \\ a_i & a_i & a_i & a_i & \cdots & a_i & 0 \end{pmatrix}$$

and every other element in the determinant has value s. For example, if $n = 11$, $r = 3$, $a_1 = a$, $m_1 = 4$, $a_2 = b$, $m_2 = 5$, $a_3 = c$, and $m_3 = 3$, then the determinant is as follows:

$$\begin{vmatrix}
0 & a & a & a & s & s & s & s & s & s & s & s & 1 \\
a & 0 & a & a & s & s & s & s & s & s & s & s & 1 \\
a & a & 0 & a & s & s & s & s & s & s & s & s & 1 \\
a & a & a & 0 & s & s & s & s & s & s & s & s & 1 \\
s & s & s & s & 0 & b & b & b & b & s & s & s & 1 \\
s & s & s & s & b & 0 & b & b & b & s & s & s & 1 \\
s & s & s & s & b & b & 0 & b & b & s & s & s & 1 \\
s & s & s & s & b & b & b & 0 & b & s & s & s & 1 \\
s & s & s & s & b & b & b & b & 0 & s & s & s & 1 \\
s & s & s & s & s & s & s & s & s & 0 & c & c & 1 \\
s & s & s & s & s & s & s & s & s & c & 0 & c & 1 \\
s & s & s & s & s & s & s & s & s & c & c & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{vmatrix}.$$

We now proceed to evaluate this determinant by applying elementary row and column operations. In each group of m_i rows ($i = 1, \ldots, r$), we subtract every row (except the last row) from the row above it. Then, in each group of m_i columns, we subtract each column (except the last column) from the column to its left. We wind up with a matrix where each square block along the diagonal has been replaced by a matrix whose diagonal entries are all $-2a_i$, (except for the lower right entry with value 0), and whose minor diagonals just below and above the main diagonal all have value a_i. Furthermore, all the s entries have disappeared with the exception of those whose rows and columns are at the end of the groups of m_i. The 1's in the last row and column have also turned to 0's except those occurring at the ends of groups of m_i entries.

In our example, the resulting determinant is

$$\begin{vmatrix}
-2a & a & & & & & & & & & & & 0 \\
a & -2a & a & & & & & & & & & & 0 \\
 & a & -2a & a & & & & & & & & & 0 \\
 & & a & 0 & & & & & s & & & s & 1 \\
 & & & & -2b & b & & & & & & & 0 \\
 & & & & b & -2b & b & & & & & & 0 \\
 & & & & & b & -2b & b & & & & & 0 \\
 & & & & & & b & -2b & b & & & & 0 \\
 & & & s & & & & b & 0 & & & s & 1 \\
 & & & & & & & & & -2c & c & & 0 \\
 & & & & & & & & & c & -2c & c & 0 \\
 & & & s & & & & & s & & c & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{vmatrix}.$$

All missing elements in the display are 0's.

In each square block, we can remove the a_i's situated along the two minor diagonals by adding in the appropriate multiple of the preceding row or column, in succession, top to bottom and left to right. In our example, we would multiply the first row by 1/2 and add it to the second row, then multiply the first column by 1/2 and add it to the second column. This leaves us with $(\frac{1}{2} - 2)a = -\frac{3}{2}a$ in row 2 column 2. Thus, we multiply row 2 by 2/3 and add it to row 3. Then we multiply column 2 by 2/3 and add it to column 3. This leaves us with $-\frac{4}{3}a$ in row 3 column 3, etc.

In general, the multipliers will be $1/2, 2/3, 3/4, \ldots (m_i - 1)/m_i$. The final numbers along the main diagonal will be $-2a_i/1, -3a_i/2, -4a_i/3, \ldots, -m_i a/(m_i - 1), (m_i - 1)a/m_i$.

In our example, we get

$$\begin{vmatrix}
\frac{-2a}{1} & & & & & & & & & & & & 0 \\
& \frac{-3a}{2} & & & & & & & & & & & 0 \\
& & \frac{-4a}{3} & & & & & & & & & & 0 \\
& & & \frac{3a}{4} & & & & & s & & & s & 1 \\
& & & & \frac{-2b}{1} & & & & & & & & 0 \\
& & & & & \frac{-3b}{2} & & & & & & & 0 \\
& & & & & & \frac{-4b}{3} & & & & & & 0 \\
& & & & & & & \frac{-5b}{4} & & & & & 0 \\
& & & s & & & & & \frac{4b}{5} & & & s & 1 \\
& & & & & & & & & \frac{-2c}{1} & & & 0 \\
& & & & & & & & & & \frac{-3c}{2} & & 0 \\
& & & s & & & & & s & & & \frac{2c}{3} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{vmatrix}.$$

Most of the entries on the major diagonal now have all 0's in their rows. We can thus expand the determinant by minors along these rows and see that the value of the determinant is

$$\prod_{i=1}^{r} (-a_i)^{m_i - 1} m_i$$

times the following determinant:

$$\begin{vmatrix}
\frac{(m_1-1)a_1}{m_1} & s & s & \cdots & s & 1 \\
s & \frac{(m_2-1)a_2}{m_2} & s & \cdots & s & 1 \\
s & s & \frac{(m_3-1)a_3}{m_3} & \cdots & s & 1 \\
\vdots & & & \ddots & & \\
s & s & s & \cdots & \frac{(m_r-1)a_r}{m_r} & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{vmatrix}.$$

This determinant is simplified by subtracting s times the bottom row from every other row. We are left with a determinant whose last row and column are all 1's (except for the 0 in the lower right corner). The remaining elements all lie along the main diagonal, and are $\frac{(m_i-1)a_i}{m_i} - s$, $i = 1, 2, \ldots, r$. In our example, this comes out to

$$\begin{vmatrix}
\frac{3a}{4} - s & 0 & 0 & 1 \\
0 & \frac{4b}{5} - s & 0 & 1 \\
0 & 0 & \frac{2c}{3} - s & 1 \\
1 & 1 & 1 & 0
\end{vmatrix}.$$

Finally, this determinant is evaluated by getting rid of the 1's in the final row. To do that, multiply each of the first r rows by the reciprocal of the diagonal element and subtract the result from the last row. This changes the 1's in the last row to 0's and changes the 0 to

$$-\sum_{i=1}^{r} \left(\frac{(m_i - 1)a_i}{m_i} - s\right)^{-1}.$$

The determinant is now upper triangular and so its value is the product of the diagonal elements. We have thus found that

$$D = \prod_{i=1}^{r} (-a_i)^{m_i - 1} \big((m_i - 1)a_i - m_i s\big) \left(- \sum_{i=1}^{r} \frac{m_i}{(m_i - 1)a_i - m_i s} \right).$$

Comparing this with formula (1) and noting that $\sum_{i=1}^{r} m_i = n+1$, we see that we can move the $(-1)^{n+1}$ to the right hand side and wind up with $(-1)^{r+1}$. Then, solving for V^2 and taking the square root of both sides proves our theorem. $\square$

Letting $r = 2$ gives us two interesting corollaries.

Corollary 14. 2.2. An n-simplex in E^n ($n \geq 1$) has one edge of length b. Every other edge has length a. Then the volume of the simplex is

$$\frac{ba^{n-2}}{n!2^{n/2}} \sqrt{2na^2 - (n-1)b^2}.$$

Corollary 14. 2.3. An n-simplex in E^n ($n \geq 1$) has every edge incident at a given vertex of length a. Every other edge has length b. Then the volume of the simplex is

$$\frac{b^{n-1}}{n!2^{n/2}} \sqrt{2na^2 - (n-1)b^2}.$$

Section 14.3.
Tetrahedra with integer sides and volume.

What is the tetrahedron of smallest (integral) volume whose edges are all integers? We conjecture that the answer is 6 as suggested by the following table which lists all Heronian tetrahedra found by computer with integer length edges and integer volume less than 25:

a	b	c	A	B	C	volume
7	6	4	2	4	5	6
8	6	4	5	7	2	6
8	7	2	4	4	6	6
8	7	3	2	4	8	6
8	7	3	6	6	4	9
8	7	5	4	4	6	15
11	10	4	8	6	8	15
12	10	7	8	5	8	15
14	8	7	10	12	8	15
14	12	8	5	7	8	15
10	8	8	6	7	4	21
11	10	8	2	8	9	21
12	11	2	8	8	10	21
12	11	2	9	10	8	21
12	11	6	9	6	8	21
13	10	6	4	8	7	21
13	10	6	8	7	8	21
14	11	4	6	8	8	21
15	14	8	8	13	8	21
7	7	6	7	7	4	24
8	7	5	6	7	5	24
9	9	6	7	7	4	24
10	9	7	4	9	7	24
11	8	7	3	8	7	24
11	10	3	7	8	7	24
11	11	4	7	7	6	24
12	7	7	4	7	7	24
12	7	7	6	7	7	24
12	9	9	4	7	7	24
12	9	9	10	7	7	24
12	11	11	6	7	7	24
14	13	2	7	8	12	24

Section 15
Conditions Implying Collinear Lattice Points

In this section, we look at some results that show when there must m collinear lattice points in a collection of lattice points.

Definition. A set, S, of lattice points is said to be *lattice-convex* if any lattice point in the convex hull of S is also in S.

In other words, in E^n, S is lattice convex if and only if

$$S = \text{hull}(S) \cap Z^n.$$

Observation 1. Let K be a convex body in E^n and let S be the set of lattice points inside K. Then S is lattice-convex.

Observation 2. If x and y are two lattice points in a lattice-convex set S, then any lattice point between x and y is also a member of S.

We have a very pretty little theorem that states when a set of lattice points must contain at least $n+1$ lattice points that lie in a row. We first state this theorem for E^2, then give alternate formulations and a generalization for E^n.

Theorem 15.1. *Let S be a lattice-convex set in the plane consisting of m^2+1 lattice points. Then S must contain some $m+1$ lattice points that are collinear.*

First we note that the set S can be a rather complicated looking set. An example is shown in figure 15-1 of 25 points that form a lattice convex set with no 6 lattice points in a row. Adding any 26th lattice point, however, (while keeping the set lattice-convex) will force some 6 lattice points to colline.

```
. . . . o o o o
. . o o o o o .
. . o o o o o .
. o o o o o . .
o o o o o . . .
. . . o . . . .
```

Figure 15-1
25 lattice points forming a
non-trivial lattice-conex set
with no 6 in a row

Proof. Consider the coordinates of the points modulo m. Since there are only m^2 distinct pairs of integers modulo m, some two of these pairs must be congruent (mod m). So suppose $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ are points in S that are congruent (mod m). That is, $x_1 \equiv x_2 \pmod{m}$ and $y_1 \equiv y_2 \pmod{m}$. Now consider the points,

$$(x_1 + \frac{x_2 - x_1}{m}k, y_1 + \frac{y_2 - y_1}{m}k)$$

as k varies from 0 to m. This is a set of $m+1$ collinear points. Furthermore, each point is a lattice point, since $m|(x_2 - x_1)$ and $m|(y_2 - y_1)$ by the congruence condition. Finally, all the $m+1$ points belong to S since the first and last ones do, and S is lattice convex. $\square$

Observation. We note that the above proof actually gives us an effective procedure for finding the $n+1$ collinear lattice points; it is not merely an existence proof.

We should also note that we didn't need the full generality of the property of lattice-convexity. We only needed the weaker condition that any lattice point between two lattice points in the set is also in the set. We give this as a formal definition.

Definition. A set of lattice points, S, is said to be *2-convex* if given any two points, x and y in S, every lattice point between x and y is also in S.

Observation 2 above shows that lattice-convexity implies 2-convexity. A quick example shows that the reverse implication does not hold. See figure 15-2.

```
. . o
o . .
. o .
```

Figure 15-2
A set that is 2-convex
but is not lattice convex

Our previous proof shows that the following stronger result is true.

Theorem 15.2. *Let S be a set of m^2+1 lattice points in the plane that is 2-convex. Then S must contain some $m+1$ lattice points that are collinear.*

The theorem also easily generalizes to higher dimensions.

Theorem 15.3. *Let S be a set of m^n+1 lattice points in E^n that is 2-convex. Then S must contain some $m+1$ lattice points that are collinear.*

The proof is analogous to the previous proof. In this case, there are at most m^n ordered n-tuples in Z^n, so some two must be congruent mod m in each coordinate. As in the previous construction, we obtain $m+1$ lattice points that are collinear beginning and ending with the two just found. By the 2-convexity, these $m+1$ points must all lie in S.

We note that the quantity m^n+1 is best possible in the above theorem, for we can always find m^n lattice points forming a lattice-convex set in which no $m+1$ colline. Namely, take the m^n lattice points inside and on the n-cube with m lattice points along each edge.

There are several other formulations of this theorem, which at first seem completely different, but are easily seen to be equivalent.

Definition. Two lattice points, x, and y, are said to form a *hole* in a set S if there is some lattice point between x and y that is not in S.

Theorem 15.4 (Ramsey Theory Formulation). *Let S be a set of m^n+1 lattice points in E^n. Then either some 2 points of S form a hole, or some $m+1$ points of S colline.*

We can view lattice points in E^n as vectors emenating from the origin. Such vectors are called lattice vectors.

Theorem 15.5 (Vector Space Formulation). *Let S be a set of m^n+1 lattice vectors in E^n. Then either there is a lattice vector, not in S, that is a convex linear combination of two lattice vectors in S or else some $m+1$ vectors in S form an arithmetic progression.*

This theorem follows from the observation that if $m+1$ vectors form an arithmetic progression, their endpoints colline.

We can also view the theorem in the light of lattice points inside convex bodies.

Theorem 15.6 (Convex Body Formulation). *Let K be a convex body in E^n containing at least $m^n + 1$ interior lattice points. Then some $m + 1$ of these lattice points must colline.*

This theorem follows immediately from Observation 1.

Appendix A
Additional Known Results

In this appendix, we list known results converning lattice points and polygons. These results hold for all convex bodies and not just for lattice polygons.

Section A.1.
Result showing that a convex set must contain a lattice point in E^2.

Theorem. *Let K be a closed convex body in the plane with area A that contains the origin O. Then K contains a non-zero lattice point if any of the following conditions hold:*

a. *(Minkowski's Theorem) K is symmetric about the origin O, and $A \geq 4$.*

b. K contains the origin, O, as an interior point, $A \geq 4$, and there exists a chord AOB of K which has midpoint O, and which partitions K into two disjoint regions having equal area.

References.

a. Lekkerkerker [68].
b. Scott [96].

Section A.2.
Results showing that a convex set must contain lattice points in E^2.

Theorem. *Let K be a closed convex body in the plane with area A, perimeter P, and diameter D. Let r and k denote positive integers. Let ϕ denotes the unique positive real number satisfying* $\sin\phi = \pi/2 - \phi$ *and let* $\lambda = 2\sqrt{2}\sin(\phi/2)$. *Then:*

a. If $A > 4.5$ and the center of gravity of K is a lattice point, then K contains at least two more lattice points.
b. If $A \geq rP/2$, then K contains r lattice points.
c. If $A \geq P$ and K is symmetric about a lattice point, then K contains at least 4 more lattice points.
d. If $A \geq P$, then K contains at least 3 lattice points.
e. If $A \geq 2^r P/2$, then K contains $2^{r+2} - 1$ lattice points.
f. If $A > r(P/2 + D)$, then K contains $2r$ lattice points.
g. If $A \geq 2^r(P/2 + D)$, then K contains $2^{r+2} - 2$ lattice points.
h. If $A > r(P/2) + (2^k - 1)D$, then K contains $2^k r$ lattice points.
i. If $A \geq 2^r(P/2) + (2^k - 1)D$, then K contains $2^k(2^{r+1} - 1)$ lattice points.
j. If $A > r\lambda D$, then K contains r lattice points in its interior.
k. If $A > r\lambda D$, then K contains r^2 lattice points in its interior.
l. If $A > r\lambda D$, then the minimum number of lattice points inside K is at most $\lfloor (2r\lambda)^2/\pi \rfloor$.
m. If $A > 4$ and K has an even number (or infinite number) of chords of symmetry through an interior lattice point then K contains another lattice point.
n. If $A > 4.5$ and K has more than one chord of symmetry through an interior lattice point then K contains another lattice point.
o. If $A > 4$ and K has more than three chords of symmetry through an interior lattice point then K contains another lattice point.

References.

a. Ehrhart [30].
b. Hammer [51].

c. Hammer [52].
d. Hammer [52].
e. Hammer [52].
f. Reich [85].
g. Reich [85].
h. Reich [85].
i. Reich [85].
j. Scott [98].
k. Hammer [54].
l. Hammer [54].
m. Arkinstall [3].
n. Arkinstall [3].
o. Arkinstall [3].

Section A.3.
Inequalites for convex sets in E^2 with no interior lattice points.

Let K be a convex set in the plane with no lattice points in its interior. Suppose K has area A, perimeter P, diameter d, (minimal) width w, inradius r, and circumradius R.

Theorem. *Let ϕ denotes the unique positive real number satisfying $\sin\phi = \pi/2 - \phi$ and let $\lambda = 2\sqrt{2}\sin(\phi/2)$. Then $A \leq \lambda D$ and this result is best possible.*

Reference. Scott [98].

Section A.4.
Results showing that a convex set must contain lattice points in E^3.

Let K be a closed convex body in E^3, with volume V.

Theorem. *If K is a closed convex solid of revolution with center of gravity at the origin O, and $V \geq 4^4/3^3$, then K contains a non-zero lattice point.*

Reference. Ehrhardt [30].

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